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THE *R*-MODE INSTABILITY IN ROTATING NEUTRON STARS

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In this review we summarize the current understanding of the gravitational-wave driven instability associated with the so-called *r*-modes in rotating neutron stars. We discuss the nature of the *r*-modes, the detailed mechanics of the instability and its potential astrophysical significance. In particular we discuss results regarding the spin-evolution of nascent neutron stars, the detectability of *r*-mode gravitational waves and mechanisms limiting the spin-rate of accreting neutron stars in binary systems.

1. Introduction

The first decade of this new millenium holds great promise for gravitational physics. The hope is that forty years of development will come to fruition as the new generation of gravitational-wave interferometers (LIGO, VIRGO, GEO600 and TAMA300) come online and reach their projected sensitivities¹. These detectors should (finally!) open a new window to the universe and make the long-heralded field of “gravitational-wave astronomy” a reality. For gravitational-wave theorists this means that decades of modelling will finally be tested by observations. The construction of the new detectors is accompanied by attempts to model all likely gravitational-wave sources in appropriate detail. This is highly relevant since reliable theoretical templates, against which one can match the noisy data-stream of the detectors, are needed if we want to extract reliable astrophysical information from future gravitational-wave data.

In this article we survey the territory where stellar pulsation theory meets gravitational-wave astrophysics. There has been a resurgence of interest in this research area (that dates back to seminal work by Thorne and his colleagues in the late 1960s^{2,3,4}) and our aim here is to provide an overview of the recent developments. We will focus our discussion on the gravitational-wave driven instability of the so-called *r*-modes. Since its serendipitous discovery a couple of years ago^{5,6}, the *r*-mode instability has attracted considerable attention. In

in this review article we aim to describe most of the recent ideas and suggestions, be they speculative or not, and put them in the appropriate context. We will try to introduce the relevant concepts and results from basic principles, in such a way that the article should be accessible to readers with little previous knowledge of this field. We feel that this is an important task since it may help us identify the many outstanding issues that must be addressed by future work.

The theory of stellar pulsation is richly endowed with interesting phenomena, and ever improving observations suggest that most stars exhibit complicated modes of oscillation. Thus it is natural to try to match theoretical models to observed data in order to extract information about the dynamics of distant stars. This interplay between observations and stellar pulsation theory is known as asteroseismology⁷. It is interesting to speculate that the advent of gravitational-wave astronomy will permit similar studies for neutron stars. One would expect most neutron stars to oscillate during various phases of their life, and actual observations would provide invaluable information about the stars internal structure and the supranuclear equation of state^{10,11}. Given their compactness, with a mass above $1.3M_{\odot}$ concentrated inside a radius of a mere 10 km, oscillating neutron stars could be interesting astrophysical sources of gravitational waves. But it is not clear that astrophysical mechanisms can excite the various oscillation modes to a detectable level. It seems likely that only the most violent processes, such as the actual formation of a neutron star following a supernova or a dramatic starquake following, for example, an internal phase-transition, will be of relevance. The strengthening evidence for magnetars^{8,9}, in which a starquake could release large amounts of energy, is also very interesting in this respect. One can estimate that these events must take place in our immediate neighbourhood (the Milky Way or the Local Group) in order to be observable.

This does not, however, rule out the possibility that neutron star pulsation may be detected by the new generation of gravitational-wave detectors. First of all, one should not discard the possibility that unique events in the life of a neutron star may excite the various modes to a relevant level. More likely to be observationally important is the long recognized possibility that various pulsation modes of rapidly rotating neutron stars may be unstable due to the emission of gravitational radiation^{12,13,14}. Should such an instability operate in a young neutron star it may lead to the emission of copious amounts of gravitational waves¹⁵. Of particular recent interest is the instability of the so-called *r*-modes, for which the gravitational waves have been estimated to be detectable for sources in the Virgo cluster (at 15-20 Mpc). If we suppose that most newly born neutron stars pass through a phase where this kind of instability is active, several such events should be observed per year once the advanced interferometers come into operation. This is a very exciting prospect, indeed.

2. Nonradial oscillations of rotating stars

A neutron star has a large number of families of pulsation modes with more or less distinct character. For the simplest stellar models, the relevant modes are high frequency pressure p -modes and the low frequency gravity g -modes⁷. For a typical nonrotating neutron star model the fundamental p -mode, whose eigenfunction has no nodes in the star, is usually referred to as the f -mode. The f -mode has frequency in the range 2-4 kHz, while the first overtone lies above 4 kHz. The g -modes depend sensitively on the internal composition and temperature distribution, but they typically have frequencies of a few hundred Hz. The standard mode-classification dates back to the seminal work of Cowling¹⁶, and is based on identifying the main restoring force that influences the fluid motion. As the stellar model is made more detailed and further restoring forces are included new families of modes come into play. For example, a neutron star model with a sizeable solid crust separating a thin ocean from a central fluid region will have g -modes associated with both the core and the ocean as well as modes associated with shearing motion in the crust^{17,18}. Of particular interest to relativists is the existence of a class of modes uniquely associated with the spacetime itself^{19,20,21}; the so-called w -modes (for gravitational wave). These modes essentially arise because the curvature of spacetime that is generated by the background density distribution can temporarily trap impinging gravitational waves. The w -modes typically have high frequencies (above 7 kHz) and damp out in a fraction of a millisecond. It is not yet clear whether one should expect these modes to be excited to an appreciable level during (say) a gravitational collapse following a supernova. One might argue that they provide a natural channel for the release of any initial deformation of the spacetime, but there are as yet no solid evidence indicating a significant level of w -mode excitation in a realistic scenario^{22,23,24}.

2.1. Linearised equations of motion

In this article we will only touch briefly on “realistic” neutron star models. Most of our discussion will concern (adiabatic) linear perturbations of relatively simple perfect fluid models. We will typically assume that the unperturbed pressure p and density ρ are related by a one-parameter equation of state, $p = p(\rho)$. A useful way of writing this is to introduce an effective polytropic index by

$$\Gamma = \frac{d \log p}{d \log \rho} \quad (1)$$

In the special case of a polytrope Γ is, of course, constant and often expressed as $\Gamma = 1 + 1/n$.

We want to consider a star rotating uniformly with frequency Ω . In the frame of reference that co-rotates with the star the equations that govern the

fluid motion (the Euler equations) can be written

$$\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + 2\vec{\Omega} \times \vec{u} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\frac{1}{\rho} \nabla p - \nabla \Phi \quad (2)$$

where \vec{u} is the fluid velocity and Φ represents the gravitational potential. Here it is worth noticing that the centrifugal term can be rewritten as

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\frac{1}{2} \nabla (\vec{\Omega} \times \vec{r})^2 \quad (3)$$

In other words, hydrostatic equilibrium corresponds to

$$\frac{1}{\rho} \nabla p = -\nabla \left[\Phi - \frac{1}{2} (\vec{\Omega} \times \vec{r})^2 \right] = -\nabla \Psi \quad (4)$$

From this we can deduce that the level surfaces

$$\Psi = \Phi - \frac{1}{2} \Omega^2 r^2 \sin^2 \theta = \text{constant} \quad (5)$$

coincide with the isobars (and isopycnic surfaces). Consequently, once we have found a convenient way to describe the level surfaces of the “effective gravitational potential” Ψ we have a complete description of our rotating fluid. In the particular case of slow rotation, the level surfaces can be described by introducing a new variable a corresponding to the mean distance of a given constant Ψ surface to the centre of mass. This new variable is related to the standard spherical coordinates (r and θ) through

$$r = a[1 + \epsilon(\theta)] . \quad (6)$$

where a ranges from 0 to R (the radius of the corresponding nonrotating star). The advantage of this representation is that the pressure and the density can now be thought of as functions of a only. For a uniform density star (which incidentally is not a bad approximation for a neutron star) we have

$$\epsilon = -\frac{5\Omega^2}{8\pi\rho} P_2(\cos \theta) \quad (7)$$

where $P_2(\cos \theta)$ is the standard Legendre polynomial, and for polytropes one can calculate the required function $\epsilon(\theta)$ from the results of Chandrasekhar and Lebovitz²⁵.

Having prescribed a rotating stellar model, we want to study small nonradial perturbations away from equilibrium. We will consequently assume that the perturbation is characterized by a displacement vector $\vec{\xi}$, which is small in a suitable sense, and yields the fluid velocity as

$$\delta \vec{u} = \partial_t \vec{\xi} \quad (8)$$

Given that the background model is stationary and symmetric with respect to the rotation axis, we can always decompose a perturbation into modes that depend on the azimuthal angle as $\exp(im\varphi)$ with m an integer. If we also assume that these modes have a harmonic time-dependence we can write

$$\vec{\xi} \rightarrow \vec{\xi} e^{i(m\varphi + \omega_r t)} \quad (9)$$

where ω_r is the oscillation frequency measured in the rotating frame. By recalling that the inertial frame is related to the rotating frame by

$$d/dt = \partial/\partial t + \vec{u} \cdot \nabla = \partial_t + \Omega \partial_\varphi \quad (10)$$

we see that the inertial frame frequency is given by

$$\omega_i = \omega_r - m\Omega . \quad (11)$$

With the Ansatz (9) the linearised Euler equations are

$$\partial_t \delta \vec{u} + 2\Omega \times \delta \vec{u} = -\omega_r^2 \vec{\xi} + 2i\omega_r \Omega \times \vec{\xi} = \frac{\delta \rho}{\rho^2} \nabla p - \frac{1}{\rho} \nabla \delta p - \nabla \delta \Phi \quad (12)$$

where $\delta \rho$ and δp represent the Eulerian perturbations in the density and pressure, respectively, and the variation in the gravitational potential is $\delta \Phi$. Note that the centrifugal force only enters this equation through its effect on the background configuration. That this should be the case is easily understood from (4): Since we are working in an Eulerian framework the position vector \vec{r} is treated as a constant.

To fully describe the perturbations we also need the equation describing the conservation of mass

$$\partial_t \delta \rho + \nabla \cdot (\rho \delta \vec{u}) = 0 \quad (13)$$

or in integrated form;

$$\delta \rho + \nabla \cdot (\rho \vec{\xi}) = 0 \quad (14)$$

as well as the perturbed Poisson equation for the gravitational field

$$\nabla^2 \delta \Phi = 4\pi G \delta \rho . \quad (15)$$

We must also prescribe an ‘‘equation of state’’ for the perturbations, i.e. a relation between the (Lagrangian) perturbations of pressure and density. Assuming adiabatic perturbations this relation is usually written

$$\frac{\Delta p}{p} = \Gamma_1 \frac{\Delta \rho}{\rho} \quad (16)$$

where Γ_1 is the adiabatic index (which need not be related to the various background quantities). In terms of the Eulerian perturbations, this corresponds to

$$\frac{\delta p}{p} = \Gamma_1 \frac{\delta \rho}{\rho} + \Gamma_1 \vec{\xi} \cdot \left[\nabla \log \rho - \frac{1}{\Gamma_1} \nabla \log p \right] \quad (17)$$

At this point it is customary to introduce the so-called Schwarzschild discriminant, \mathcal{A}_s . It corresponds to the magnitude of the vector in the square bracket above, and since p and ρ depend only on a in our rotating model we have

$$\mathcal{A}_s = \frac{1}{\rho} \frac{d\rho}{da} - \frac{1}{\Gamma_1 p} \frac{dp}{da} \quad (18)$$

As we will see later, the Schwarzschild discriminant plays a crucial role in stellar pulsation theory. Physically, radial variation in \mathcal{A}_s corresponds to internal composition or temperature gradients in the star. In particular, the special case $\mathcal{A}_s = 0$, which leads to the perturbations obeying the same equation of state as the background fluid ($\Gamma_1 = \Gamma$), is often used for neutron stars. In the following, we will refer to such perturbations as being “isentropic”. The motivation for this assumption is that the temperature of most neutron stars is far below the relevant Fermi temperature, and they can consequently be considered as essentially having zero temperature.*

2.2. Rotational effects on pulsation modes

We are now well equipped to discuss the effect that rotation has on the various pulsation modes of a star. To understand this issue it is helpful to recall how a mode calculation proceeds in the non-rotating case. For spherically symmetric stars one can separate the pulsations into two general classes. We will refer to these as spheroidal and toroidal perturbations. In relativistic studies these two classes are often called polar and axial perturbations (or even and odd parity perturbations), respectively.

Spheroidal modes have displacement vectors of form

$$\frac{\vec{\xi}}{r} = \sum_{lm} \left(S_{lm}, H_{lm} \partial_\theta, \frac{H_{lm}}{\sin \theta} \partial_\varphi \right) Y_{lm} \quad (19)$$

where $Y_{lm}(\theta, \varphi)$ are the standard spherical harmonics. Spheroidal perturbations are accompanied by variations in the pressure and the density which can, since they are scalar quantities, always be expanded in terms of the spherical harmonics. Hence, we have

$$\delta p = \sum_{lm} \delta p_{lm} Y_{lm} \quad (20)$$

and similar for $\delta\rho$. By analyzing the equations for a spheroidal perturbation one can deduce that a simple stellar model allows two distinct classes of pulsation modes (cf. the monograph by Unno et al.⁷ for details). These have frequencies

*However, it is worth pointing out that realistic neutron stars may still have $\mathcal{A}_s \neq 0$ because of radial variation in the chemical composition (say, a varying proton fraction in the core)²⁶. In view of this, whenever we refer to the “isentropic” case in this article, we intend this to mean stars with *no internal entropy or composition gradients*.

that are governed by

$$\omega^2 \approx \frac{l(l+1)c_s^2}{r^2} \quad (21)$$

where $c_s^2 = \Delta p / \Delta \rho$ is the sound speed, and

$$\omega^2 \approx -g\xi_r \mathcal{A}_s \quad (22)$$

where $g = |dp/dr|/\rho$ is the local gravitational acceleration. These classes of modes are known as the p - and the g -modes, respectively.

Toroidal modes have eigenvectors that can be written

$$\frac{\vec{\xi}}{r} = \sum_{lm} \left(0, \frac{T_{lm}}{\sin \theta} \partial_\varphi, -T_{lm} \partial_\theta \right) Y_{lm} = \sum_{lm} \frac{1}{\sqrt{l(l+1)}} T_{lm} \vec{Y}_{lm}^B \quad (23)$$

where \vec{Y}_{lm}^B are the magnetic multipoles introduced by Thorne²⁷. A non-rotating perfect fluid model has toroidal modes with non-zero frequency, but in a model with a solid crust there are distinct toroidal shear modes^{17,18}. It should also be mentioned that the relativistic theory yields both toroidal and spheroidal w -modes²¹.

Because of the symmetry of the non-rotating problem, modes corresponding to different l and m decouple. In fact, it is sufficient to consider the $m = 0$ case since the non-zero m solutions then follow after a trivial rotation. Hence, there is no need to sum over the various l and m in the case of perturbed spherical stars. The case of rotating stars is much more complicated. First of all, the symmetry is broken in such a way that the various $-l \leq m \leq l$ modes become distinct. As a first approximation one finds that⁷

$$\omega_i(\Omega) = \omega_r(\Omega = 0) - m\Omega(1 + C_{lm}) + O(\Omega^2) \quad (24)$$

according to an inertial observer. Here, C_{lm} is a function that depends on the mode-eigenfunction in a non-rotating star. Secondly, rotation has a significant effect on the eigenfunctions by coupling the various l -multipoles. As the rotation rate is increased an increasing number of Y_{lm} 's are needed to describe a mode. One must also account for coupling between the spheroidal and toroidal vectors. To illustrate this, let us consider a toroidal mode that corresponds to a multipole $[l, m]$ in the $\Omega = 0$ limit. After including the first rotational correction the eigenfunction of this mode becomes

$$\frac{\vec{\xi}}{a} = \left(0, \frac{T_{lm}}{\sin \theta} \partial_\varphi, -T_{lm} \partial_\theta \right) Y_{lm} + \sum_{\nu=l\pm 1} \left(S_{\nu m}, H_{\nu m} \partial_\theta, \frac{H_{\nu m}}{\sin \theta} \partial_\varphi \right) Y_{\nu m} \quad (25)$$

In other words, the $[l, m]$ toroidal mode couples to the $[l \pm 1, m]$ spheroidal ones. The situation is analogous for rotationally modified spheroidal modes.

2.3. Low-frequency modes

A rotating star has two sets of low-frequency modes, the spheroidal *g*-modes and the toroidal *r*-modes^{28,29,30,31,32}. To understand the origin and nature of these modes it is helpful to digress on the perturbations of non-rotating (spherical) stars. A spherical star has a degenerate spectrum of zero-frequency modes. These modes correspond to neutral convective currents, and the nature of the corresponding fluid perturbations can be understood in the following way: As follows from (12) and (13) stationary, isentropic, perturbations of a spherical star are solutions to

$$\nabla \cdot (\rho \delta \vec{u}) = 0 \quad (26)$$

and

$$\frac{\delta \rho}{\rho^2} \nabla p - \frac{1}{\rho} \nabla \delta p - \nabla \delta \Phi = 0 \quad (27)$$

Since these two equations decouple we can consider any solution as a superposition of two distinct classes of solutions³⁶:

$$\begin{aligned} \delta \vec{u} &\neq 0, & \delta p = \delta \rho = \delta \Phi = 0 \\ \delta \vec{u} &= 0, & \delta p, \delta \rho, \delta \Phi \text{ nonzero} \end{aligned}$$

Since any static self-gravitating perfect fluid must be spherical the second type of solution simply identifies a neighbouring equilibrium model. Hence, all stationary nonradial perturbations of a spherical star must be of the first kind.

What can we deduce about the fluid velocity for these stationary modes? First of all, one can readily verify that all toroidal displacements satisfy (26). In other words, the function T_{lm} is unconstrained. In the case of spheroidal perturbations the situation is slightly different, and one finds that the following relation between S_{lm} and H_{lm} must be satisfied;

$$\frac{d}{dr}(\rho r^3 S_{lm}) - l(l+1)\rho r^2 H_{lm} = 0 \quad (28)$$

Still, one of the two functions is left unspecified also in this case.

Before moving on to rotating stars, we need to consider also the case of non-isentropic perturbations. Then (17) provides a relation between the radial component of $\vec{\xi}$ and δp and $\delta \rho$. This will obviously not affect the above conclusion for toroidal modes: There is still a space of zero-frequency toroidal modes in a non-isentropic star. But the stratification associated with a non-zero \mathcal{A}_s breaks the degeneracy in the spheroidal case, and gives rise to distinct *g*-modes, cf. (22).

Having understood the nature of the zero-frequency modes in the $\Omega = 0$ limit, it is easy to see what happens in the rotating case. For spinning stars the degeneracy of the zero-frequency modes is broken. The Coriolis force provides a weak restoring force that gives the toroidal modes genuine dynamics^{7,30}. This leads to the so-called *r*-modes. The main properties of the *r*-modes can be understood by carrying out a simple exercise. Assume that the mode is purely

toroidal to leading order also in the rotating case. Then the motion is essentially horizontal and ξ_r , $\delta\rho$ and δp are of higher order in Ω . Now consider the radial component of the equation we get by taking the curl of (12);

$$\partial_t(\nabla \times \delta\vec{u})_r + 2\nabla \times (\vec{\Omega} \times \delta\vec{u}) = \partial_t(\nabla \times \delta\vec{u})_r + \delta u_h \cdot \nabla_h(2\Omega_r) = 0 \quad (29)$$

In this equation subscripts r and h represent radial and horizontal components, respectively, and we have neglected terms of order Ω^2 . We now translate this result into the inertial frame and get

$$\frac{d}{dt} \left[(\nabla \times \delta\vec{u}) + 2\vec{\Omega} \right]_r = 0 \quad (30)$$

Here we can identify the quantity in the square bracket as the radial component of the total vorticity. This result thus shows that the radial component of the vorticity is conserved by the r -mode motion³⁰. Furthermore, by inserting the toroidal eigenvector in (29) we readily find a first approximation of the r -mode eigenfrequency

$$\omega_r \approx \frac{2m\Omega}{l(l+1)} . \quad (31)$$

At this level of approximation different radial shells in the star are effectively uncoupled. Hence, the radial dependency of the r -mode eigenfunctions remain undetermined. To determine the radial behaviour of the modes we need to extend the analysis to order Ω^2 . The reason for this can be understood as follows: From the linearised equations we can see that our leading order toroidal perturbation will not couple to the density and pressure perturbations until at order Ω^2 . Indeed, if we write down the equations that describe terms of order Ω we find that they decouple completely from the leading order toroidal perturbation. To linear order in Ω the r -mode assumption is therefore consistent with $\delta p = \delta\rho = \delta\Phi = \xi_r = 0$. As far as the higher order corrections to the r -mode frequency are concerned, one can argue that the next non-trivial contribution must be of order Ω^3 . Essentially, the frequency correction should have the same effect relative to the leading order frequency irrespective of the sense of rotation of the star. It cannot be that the r -mode frequency is different in magnitude for positive and negative Ω . If that were the case the frequency would depend on the orientation of the observer, and change if he/she (say) decided to stand on his/her head, which clearly cannot make sense. Thus we anticipate that the r -mode frequency can be written in terms of a dimensionless eigenvalue ω_2 as

$$\omega_r = \frac{2m\Omega}{l(l+1)} \left[1 - \omega_2 \frac{R^3\Omega^2}{M} \right] . \quad (32)$$

We also expect that the r -mode eigenfunctions will take the form (25) with $T_{lm} = O(1)$ and $S_{l\pm 1m} \sim H_{l\pm 1m} \sim O(\Omega^2)$, and that the pressure and density variations will both be of order Ω^2 .

Table 1. Calculated r -mode frequency corrections ω_2 for constant density stars ($n = 0$) and $n = 1$ polytropes. Results are shown both for the full calculation (nC) and the Cowling approximation (C). The results are for isentropic perturbations and the fundamental $l = m$ modes.

| l | $\omega_2(n = 0)$ | | $\omega_2(n = 1)$ | |
|-----|-------------------|-------|-------------------|-------|
| | nC | C | nC | C |
| 2 | 0.765 | 0.913 | 0.398 | 0.453 |
| 3 | 0.797 | 0.844 | 0.427 | 0.443 |
| 4 | 0.730 | 0.749 | 0.399 | 0.405 |

Once we have cranked through the algebra, the following picture emerges: For adiabatic non-isentropic perturbations, we have a Sturm-Liouville problem²⁹ for the function T_{lm} . This means that there will be an infinite set of r -modes for each combination of l and m . These modes are suitably labeled by the number of radial nodes in the respective eigenfunctions. A word of caution is in order, however. It is clear that we can reliably determine only a limited number of r -modes within the slow-rotation scheme: At some point the frequency correction will become large, and the assumption that it is a small perturbation will be violated. From (32) we can see that the scheme breaks down unless

$$|\omega_2| \ll \frac{M}{R^3 \Omega^2} \quad (33)$$

Interestingly, the r -mode results for the isentropic case are rather different. When $\mathcal{A}_s = 0$ one can determine the eigenfunctions already at lowest order in the calculation²⁹. Formally the calculation is taken to order Ω^2 , but when $\mathcal{A}_s = 0$ one can combine the r and φ components of (12) to provide an equation relating the leading order contributions to ξ_θ and ξ_φ . This yields a rather simple equation for T_{lm} :

$$\left[aT_{lm} + \frac{1}{l(l+1)} \frac{d}{da} (a^2 T_{lm}) \right] \sin \theta \frac{dP_{lm}}{d\theta} - \frac{d}{da} (a^2 T_{lm}) \cos \theta P_{lm} = 0 \quad (34)$$

Using standard relations for the Legendre functions one can show that this implies that we must have $T_{lm} = 0$ unless $l = m$. In other words, in the isentropic case r -modes can only exist for $l = m$, and in that case the above equation readily yields solutions of form

$$T_{ll} \sim a^{l-1} . \quad (35)$$

If we proceed to a full order Ω^2 mode calculation^{29,30,38,39,40,37}, we find the frequency corrections listed in Table 1.

How can we understand the radical difference between the isentropic and the non-isentropic cases? A more detailed study of the r -mode eigenfunctions in the non-isentropic case shows that the fundamental mode (which has no nodes in the radial eigenfunctions) is well described by (35) also when $\mathcal{A}_s \neq 0$. Furthermore, the eigenfrequency of this mode remains virtually unchanged³⁸ as

$\mathcal{A}_s \rightarrow 0$. Meanwhile, the frequency corrections for all the other r -modes change dramatically. This suggests the following explanation: A non-isentropic star has an infinite number of r -modes for each l and $m \neq 0$. These modes are distinct because of the stratification associated with the radial variation of \mathcal{A}_s . In this sense these modes are close relatives to the spheroidal g -modes. The single r -mode that remains for $l = m$ in an isentropic star is likely associated with the fact that the star is a sphere and not a cylinder³⁸ (which is relevant since the Coriolis operator has cylindrical symmetry). This “stratification” leads to the existence of a unique mode also in the isentropic case and explains why this mode is weakly dependent on variations in \mathcal{A}_s . Only one mystery remains: What happens to all the other r -modes as the star becomes isentropic? We will return to this question in section 7.1.

Detailed calculations show that the r -modes are such that $\Delta p \ll \delta p$ also to order Ω^2 . This means that the modes are well confined to the potential surfaces of the rotating configuration. In principle, this means that one would expect these modes to be associated with small variations in the gravitational potential. Given this, and the fact that the modes are more or less localized in the low density surface region of the star, the Cowling approximation (wherein $\delta\Phi$ is neglected) should be accurate for the r -modes. Calculations have verified that the modes are obtained to within a few percent in the Cowling approximation^{29,30}. This is illustrated in Table 1.

What do the r -mode results imply for rapidly spinning neutron stars? A stable star can never spin faster than the rate at which matter is ejected from the equator. This corresponds to the Kepler limit, which is well approximated by

$$\Omega_K \approx \frac{2}{3} \sqrt{\pi G \bar{\rho}} \quad (36)$$

where $\bar{\rho}$ represents the average density of the corresponding nonrotating star. We can compare this estimate to an empirical formula, based on fully relativistic calculations for stellar models using realistic equations of state, which suggests that^{33,34,35}

$$\Omega_K \approx 0.78 \sqrt{\pi G \bar{\rho}} \quad (37)$$

Typically, the Kepler limit corresponds to a rotation period in the range¹⁴ 0.5 – 2 ms. It has been shown that $O(\Omega^2)$ slow-rotation stellar models are accurate to within a few percent compared to full numerical solutions. Consequently, one would expect a slow-rotation mode-calculation to be reasonably accurate even for the fastest spinning neutron stars. From (33) we immediately see that an r -mode solution ought to be reliable as long as

$$|\omega_2| \ll 5M_{1.4}R_{10}^3P_{-3}^2. \quad (38)$$

We have parameterized this formula in terms of canonical neutron star values:

$$M_{1.4} = M/1.4M_\odot \quad (39)$$

$$R_{10} = R/10 \text{ km} \quad (40)$$

$$P_{-3} = P/1 \text{ ms} \quad (41)$$

Given this and the data in Table 1 we see that the fundamental *r*-mode as calculated in the slow-rotation approximation may be reliable also for the fastest spinning pulsars.

2.4. *Digression: The fluid motion*

Before we turn our attention to the interplay between *r*-modes, gravitational radiation and viscosity, it is relevant to discuss various features of the modes in some more detail.

An important concept in the study of pulsation modes of rotating stars is the “pattern speed” of the mode. Given that every mode is proportional to $e^{i(m\varphi + \omega t)}$ we can see that, holding a and θ fixed, surfaces of constant phase are described by

$$m\varphi + \omega t = \text{constant} \quad (42)$$

After differentiation this leads to

$$\frac{d\varphi}{dt} = -\frac{\omega}{m} = \sigma \quad (43)$$

which defines the pattern speed σ of the mode.

Having defined this concept, we make two observations. First of all we see that the pattern speed for a typical *r*-mode is

$$\sigma_r = -\frac{2\Omega}{l(l+1)} \quad (44)$$

according to an observer rotating with the star. On the other hand, an inertial observer would find

$$\sigma_i = \Omega \frac{(l-1)(l+2)}{l(l+1)}. \quad (45)$$

That is, although the modes appear retrograde in the rotating system an inertial observer would view them as prograde.

Another useful result concerns the *p*-modes. Calculations for nonrotating stars show that the frequency of these modes increases (for a given overtone) with m slower than linearly. This means that as we increase m , the pattern speed of the *p*-modes will decrease, cf. (43). As a consequence, even though the high order *p*-modes have arbitrarily large frequencies, one can always find a mode with a very small pattern speed. The implications of this result will become clear in the following section.

Let us now return to the *r*-modes and focus on the detailed fluid motion. We will concentrate on the $l = m$ *r*-modes of isentropic stars. Introducing a suitable

dimensionless amplitude α , we can write the r -mode velocity field (measured by an observer that is co-rotating with the star) as¹⁵

$$\vec{u} \approx \alpha \Omega R \left(\frac{a}{R} \right)^l \vec{Y}_l^B e^{i\omega_r t} \quad (46)$$

We can readily illustrate this mode by drawing this vector field on the two-sphere, cf. Figure 1 and Figure 1 of Saio³⁰. This gives us an idea of the overall mode pattern. Every half period of oscillation the motion changes direction, and as we have already deduced, the pattern drifts backwards with respect to the sense of rotation (according to an observer on the star).

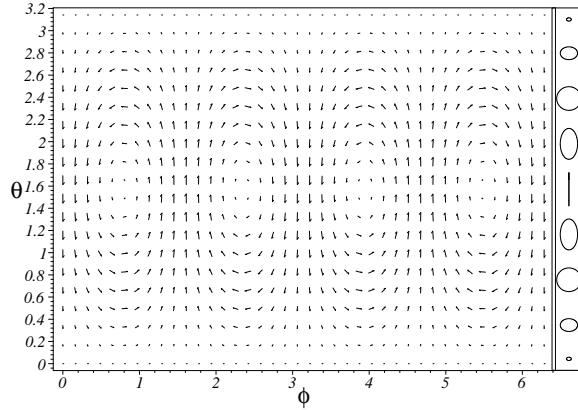


Fig. 1. The large left frame shows the velocity field of the $l = m = 2$ r -mode at an instant in time. The narrow frame on the right is a qualitative picture of the actual motion of individual fluid elements (to leading order in the slow-rotation expansion).

The velocity field in Figure 1 does not, however, give us a good picture of the motion of the individual fluid elements. To understand the nature of the r -modes better we consider fluid elements on the surface of the star. We can deduce from (46) that each fluid element moves according to

$$\delta r \approx 0, \quad (47)$$

$$\delta\theta = \frac{\xi_\theta}{R} \propto \alpha \sin \theta \cos[m\varphi + \omega_r t], \quad (48)$$

$$\delta\varphi = \frac{\xi_\varphi}{R} \propto \alpha \sin \theta \cos \theta \sin[m\varphi + \omega_r t]. \quad (49)$$

From these equations we see that

$$\frac{\delta\theta^2}{\sin^2 \theta} + \frac{\delta\varphi^2}{\sin^2 \theta \cos^2 \theta} \propto \alpha^2 \quad (50)$$

In other words, to leading order each fluid element moves on an ellipse the size of which is determined by the mode-amplitude α . As $\theta \rightarrow 0$ or π the

ellipses approach circles and the radius shrinks to zero. We can also infer that a fluid element on the equator only moves up and down in the θ direction. These features are illustrated in Figure 1. This is obviously only a first approximation, and one would expect higher order terms (both in Ω and α) to be important for rapidly spinning stars. We will return to briefly discuss the fluid motion and possible higher order secular effects in Sections 7.3-7.4.

It is also worth trying to gain some understanding of the mode-amplitude α . Since we are working within a perturbative approach we naturally require that α be small, but what exactly does this mean? A useful measure is provided by the radial displacement. We know that the radial displacement is of order Ω^2 , and we can approximate it using (46) together with equations (12)–(14) from Kokkotas and Stergioulas³⁹. In particular, if we consider a fluid element located on the surface of the star, we find that

$$\frac{\delta r}{R} \approx 0.07\alpha \left(\frac{\Omega}{\Omega_K} \right)^2 \quad (51)$$

This estimates the “height” of the surface waves induced by the r -modes. Later on, when we discuss the possible effect that the unstable r -modes may have on the spin of a newly born neutron star, we will consider values of α of the order unity. Given that a neutron star has a radius of roughly 10 km, such values of α are obviously by no means “small”, as the correspond to waves of amplitude 700 m on the surface of a star spinning at the Kepler limit. This is a very important insight, since it means that our discussion must be considered as (at best) an extrapolation into a regime where nonlinear effects should be relevant.

Finally, it is worth emphasising that even though the r -modes are relatively exotic to astrophysicists they are very familiar to oceanographers and meteorologists. The r -modes are analogous to the so-called Rossby waves in the Earth’s atmosphere and oceans. These waves were first discussed by the Swedish meteorologist Carl-Gustaf Rossby, and it is the association with Rossby waves that gives the r -modes their name. There are many interesting discussions of terrestrial Rossby waves in the literature. We should try to learn as much as possible from our colleagues in, for example, geophysics. Good starting points are the monographs by Greenspan⁴¹ and Pedlosky⁴².

3. Gravitational-wave instabilities in rotating stars

3.1. The CFS mechanism

Many different instabilities can operate in a neutron star. The most familiar instability is due to the existence of a maximum mass (beyond which the star must collapse to form a black hole). Furthermore, above Ω_K there are no equilibrium states (unless the star rotates differentially). Thus the star must shed some of its angular momentum before it can settle down into a stationary

state. There have been many studies of this process in the literature^{43,44,45}. The so-called bar-mode instability grows essentially on the dynamical timescale of the star, and leads to gravitational waves that may well prove to be observable.

It has long been recognized that the addition of various dissipation mechanisms may lead to secular instabilities, proceeding on a relatively long timescale depending on the magnitude of the added effect. The archetypal secular instability is associated with viscosity. This instability was first studied by Roberts and Stewartson⁴⁶ (for more recent results, see Lindblom⁴⁷). Essentially, viscosity admits a transition to a lower energy state by violating the conservation of circulation (which holds in a perfect fluid star). In a similar way, rotating neutron stars are generically unstable due to the emission of gravitational waves (which leads to angular momentum of the star not being conserved). That such an instability would operate was first established by Chandrasekhar⁴⁸ for the Maclaurin spheroids. This interesting result was subsequently put on a rigorous footing by Friedman and Schutz^{49,50,51}, who also proved that the instability is generic: All rotating perfect fluid neutron stars are unstable!

The mechanism for gravitational-wave instability can be understood in the following way^{49,50}. Consider first a non-rotating star. Then the mode-problem leads to eigenvalues for ω^2 , cf. Unno et al⁷, which in turn gives equal values $\pm|\omega|$ for the forwards and backwards propagating modes (corresponding to $m = \pm|m|$). These two branches of modes are affected by rotation in different ways, cf. (24). A backwards moving mode will be dragged forwards by the stellar rotation, and if the star spins sufficiently fast the mode will move forwards with respect to the inertial frame. Meanwhile, the mode is still moving backwards in the rotating frame. The gravitational waves from such a mode carry positive angular momentum away from the star, but since the perturbed fluid actually rotates slower than it would in absence of the perturbation the angular momentum of the retrograde mode is negative. The emission of gravitational waves consequently makes the angular momentum of the mode increasingly negative and leads to an instability. This class of frame-dragging instabilities is usually referred to as Chandrasekhar-Friedman-Schutz (CFS) instabilities. It is easy to see that the CFS mechanism is not unique to gravitational radiation. Any radiative mechanism will do, and even though this possibility has not yet attracted much attention, one would expect a similar instability to exist also for electromagnetic waves (see ¹³¹ and comments in section 7.3).

The fact that the emission of gravitational radiation causes a growth in the mode energy in the rotating frame, despite the decrease in the inertial frame energy, may at first seem a bit strange. However, it can be understood from the relation between the two energies:

$$E_r = E_i - \Omega J . \quad (52)$$

From this we see that E_r may increase if both E_i and J decrease^{49,50}. In other words, when the mode radiates away angular momentum the star can find a

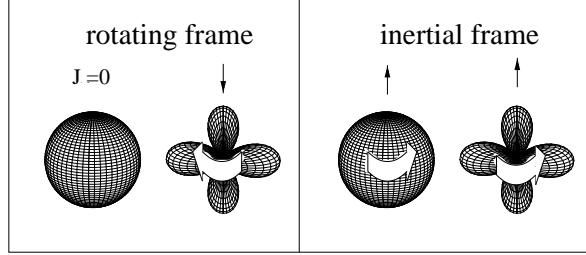


Fig. 2. A schematic illustration of the conditions under which the CFS instability is operating. A perturbed star can be viewed as a superposition of a uniformly rotating background and a nonaxisymmetric perturbation. A mode is unstable if it is retrograde according to an observer in the fluid (left), but appears prograde in the inertial frame (right).

rotational state of lower angular momentum *and* lower energy. Under these conditions the mode amplitude may grow.

Let us take a closer look at some of the relevant ideas for this mechanism. The Friedman-Schutz criterion for instability relies on the so-called canonical energy E_c being negative. The canonical energy is defined as

$$E_c = \frac{1}{2} \int \left[\rho |\partial_t \vec{\xi}|^2 - \rho |\vec{u} \cdot \nabla \vec{\xi}|^2 + \Gamma_1 p |\nabla \cdot \vec{\xi}|^2 + \vec{\xi}^* \cdot \nabla p \nabla \cdot \vec{\xi} + \vec{\xi} \cdot \nabla p \nabla \cdot \vec{\xi}^* + \xi^{i*} \xi^{j*} (\nabla_i \nabla_j p + \rho \nabla_i \nabla_j \Phi) - \frac{1}{4\pi G} |\nabla \delta \Phi|^2 \right] dV \quad (53)$$

which is a conserved quantity.

Similarly, we can define another conserved quantity

$$J_c = -\text{Re} \int \rho \partial_\varphi \xi^{i*} (\partial_t \xi_i + \vec{u} \cdot \nabla \xi_i) dV \quad (54)$$

i.e. a canonical angular momentum. If E_c is negative at the outset and the system (the star) is coupled to another system (the radiation) in such a way E_c must decrease with time, then the absolute value of E_c will increase and the associated mode is unstable. Generally, an instability can be distinguished in a mode-independent way by constructing (canonical) initial data $[\vec{\xi}, \partial_t \vec{\xi}]$ such that E_c is negative. However, to do this is not at all trivial. From the computational point of view the most straightforward task is to find the so-called neutral modes (which have zero frequency in the inertial frame), that signal the onset of instability. Furthermore, if we manage to proceed further and calculate a general mode of a rotating star, we can readily evaluate E_c to assess its stability. As was shown by Friedman and Schutz^{49,50}, all mode solutions can be viewed as canonical data, and therefore it is sufficient to show that the displacement vector associated with the mode leads to $E_c < 0$ to demonstrate the presence

of an instability. One can show that the condition $E_c < 0$ is equivalent to the simple notion of a retrograde mode being dragged forwards, i.e. a change of sign in the pattern speed as viewed in the inertial frame. This important conclusion follows immediately from the relation

$$E_c = -\frac{\omega_i}{m} J_c = \sigma_i J_c \quad (55)$$

which is a general property of linear waves. Clearly, E_c changes sign when the pattern speed σ_i passes through zero.

It is interesting at this point to contrast the gravitational-wave driven instability to that due to viscosity. For uniformly rotating stars one can show that the combination

$$\delta E - \Omega \delta J = E_c - \Omega J_c = -\frac{\omega_r}{m} J_c = \sigma_r J_c = E_{c,R} \quad (56)$$

relating the first order changes in the kinetic energy and angular momentum to a mode-solution, is gauge-invariant. $E_{c,R}$ can be viewed as the canonical energy in the rotating reference frame. Viscosity leads to $E_{c,R}$ being a decreasing function of time. Comparing (55) to (56) we see that the onset of the viscosity driven instability is signalled by the vanishing of the pattern speed in the rotating frame ($\sigma_r = 0$).

3.2. Estimates for the *f*-mode instability

Many families of neutron star oscillation modes contain members that, for a sufficiently high rotation rate, satisfy the CFS instability criterion. We have already pointed out that the large m *p*-modes may have arbitrarily small pattern speeds in a spherical star. Combining this result with the qualitative formula (24) we can deduce that these *p*-modes will become unstable already at relatively slow rotation rates. Because it has no nodes in its radial eigenfunctions the fundamental *p*-mode (the *f*-mode) radiates gravitational waves more efficiently than the various overtones, and consequently leads to the strongest instability for any given m .

So why is it that, if all rotating stars are generically unstable, we observe millisecond pulsars? This is a crucial question, the answer to which follows from several important results. First of all, we have seen that the large m modes become unstable at the lowest rates of rotation. But as Comins^{52,53} have shown (for the Maclaurin spheroids) the growth rate of these modes decreases exponentially with m . In order to play an astrophysical role a mode must grow fast compared to the evolutionary timescale of the star. In practice, this means that one would not expect modes with m significantly larger than (say) 10 to be relevant. Secondly, it turns out that gravitational radiation and viscosity compete. Viscosity tends to suppress the CFS instability, and since it operates locally on short length scales it is more effective for high m modes. Detailed

calculations^{54,55} have shown that viscosity suppresses the CFS instability in the *p*-modes for $m > 5$.

To establish a reliable consensus regarding the stability properties of rotating neutron stars is a high priority issue. However, despite a considerable amount of work being devoted to the task this subject may still hide many of its secrets. Given that a general criterion for stability is outstanding, we must attempt to calculate all the various modes of a rotating star to see if any of them are unstable. This is a formidable task, but considerable progress has been made. Currently, most of our understanding of unstable modes comes from Newtonian calculations, in particular for the Maclaurin spheroids.

Until recently, the conventional wisdom was that the *f/p*-modes would lead to the strongest gravitational-wave instability. This notion dates back to the very first investigations of the CFS mechanism. Friedman and Schutz⁵⁰ suggested that “we can probably safely conjecture that the *p*-modes of nonisentropic stars are the most important ones for stability, since they typically involve larger density changes than the *g*-modes do, and so will radiate gravitational waves more effectively”. This would seem to be a reasonable assumption, and it explains why studies of the CFS instability in neutron stars remained focussed on the *f/p*-modes for almost twenty years.

Before proceeding to discuss the new ideas regarding the CFS-instability that have emerged in the last couple of years, we will summarise the status of the field as of a few years ago. The current understanding of the instability in the *f*-modes of a rotating neutron star is essentially based on two different bodies of work.

Newtonian results for the *f*-modes of rapidly rotating stars can be used to estimate the competing influences of gravitational radiation reaction and various viscosities. This leads to a prediction that the $m = 4$ mode provides the most stringent limit on rotation, and that the CFS instability would limit the rotation of a normal fluid star to⁵⁴ $\Omega > 0.95\Omega_K$. Similar estimates for superfluid stars, suggest that dissipation due to the so-called mutual friction (see section 4.5) will completely suppress the instability⁷⁶.

A fully relativistic calculation of pulsation modes of rotating stars is still outstanding. The best results so far concern neutral (zero frequency in the inertial frame) modes that would signal the onset of instability⁵⁷, and modes calculated within the Cowling approximation^{58,59}. As could perhaps have been anticipated, these results indicate that relativistic effects strengthen the instability. For example, while Newtonian calculations predicted that the $m = 2$ *f*-mode does not become unstable for $\Omega < \Omega_K$ the relativistic results show that this mode can in fact become unstable and that it may provide the strongest constraint on the rotation rate of a relativistic star.

4. The *r*-mode instability

The discovery that the r -modes of a rotating neutron star are generically unstable due to the emission of gravitational waves^{5,6} may have come as a slight surprise, but in retrospect the result is rather obvious. The real surprise is that this was not realized earlier. After all, the r -modes have been discussed in Newtonian stellar pulsation theory for the last twenty years or so (following a pioneering 1978 paper by Papaloizou and Pringle²⁸), and their properties are described in monographs on the subject⁷. Since the modes are prograde in the inertial frame and retrograde in the co-rotating frame, cf. (44)-(45), they satisfy the Newtonian criterion for the CFS-instability. Hence, one can easily deduce that the r -modes ought to be unstable. That this instability is generic also in the relativistic case was first shown by Friedman and Morsink⁶, who proved that there exist toroidal initial data for which the canonical energy E_c is negative.

In perfect fluid stars the r -modes are unstable at all rates of rotation⁵. This is an important observation, because it indicates that the instability may well be astrophysically relevant even though the r -modes do not lead to large variations in the density and therefore would not intuitively be associated with strong gravitational waves. The reason one might expect the r -mode instability to be competitive with the more familiar instability in the f -modes is that an f -mode has nonzero frequency in a spherical star. Thus, the star must be spun up to a critical rotation rate (Ω_c) in order for the f -mode to go unstable, cf. (24). Because the r -modes are unstable at all rates of rotation

their growth times may become rather short already at spin rates below Ω_c , which could then lead to the r -mode instability being stronger than the f -mode one. The first attempt to calculate the r -mode growth times⁵ suggested that this would be the case. The growth times for the r -modes were estimated to be very short indeed. It was suggested that the amplitude of an unstable r -mode would grow by a factor of two in a second for a typical neutron star spinning with a period of 1.6 ms. However, these estimates came with a serious disclaimer: The method used to determine the modes, and infer the growth times, was not designed to study low-frequency modes.

Nevertheless, these first estimates provided a strong motivation for more detailed work. As a first step towards understanding this new instability better it seemed sensible to obtain estimates that allowed an immediate comparison with the established f -mode results (recall that these estimates suggest that the f -mode instability becomes active at rotation rates above $0.95\Omega_K$). Such a comparison would establish whether the r -mode instability is of relevance astrophysically or if it is just a peculiarity associated with general relativity. To make the desired comparison one must calculate the characteristic growth time of the modes, as well as the damping times due to various dissipation mechanisms that may be relevant. As in the case of the f -mode, the stability of the r -modes depends on the competing influences of gravitational radiation reaction (that drives the mode) and viscous damping. In order to establish the astrophysical relevance of the instability we must confirm two things: First, the

unstable mode must grow on a timescale that is astrophysically “short”, for example, significantly below the age of the universe. Secondly, the instability must be able to win the tug-of-war against all relevant dissipation mechanisms.

In this section we will discuss various estimates that have been made, especially for damping mechanisms that may counteract the growth of an unstable *r*-mode. The estimates we present are made in the framework of Newtonian gravity. Since the *r*-mode instability is a truly relativistic effect this obviously makes these results somewhat ad hoc. But we must remember that there are as yet no fully relativistic calculations of pulsation modes of rapidly rotating neutron stars. In absence of such results we expect Newtonian estimates to provide useful indications. One would not expect more detailed models, eg. in full general relativity, to completely change the Newtonian picture (we comment on some recent results regarding relativistic *r*-modes in section 7.2). After all, the relativistic corrections are likely to be of the order of (say) 20% and there are far larger uncertainties associated with other aspects of this problem (eg. the microphysics).

We will adopt the standard strategy for estimating instability timescales⁵⁴. First we assume that the true mode-solution is well represented by the solution to the non-dissipative perturbation equations (as described in Section 2.3). Then we use these solutions to evaluate the effect of the various dissipation mechanisms and add their respective contributions to the rate of change of the mode energy dE/dt . Finally, we can verify that the first assumption is justified by checking that the estimated growth/damping time is considerably longer than the oscillation period of the mode.

We estimate the timescale t_d of each dissipation mechanism by assuming that the eigenfunctions are proportional to $\exp(t/t_d)$. Then we recall that the mode-energy follows from the square of the perturbation and deduce that

$$\frac{dE}{dt} = -\frac{2E}{t_d} \quad (57)$$

where

$$E \approx \frac{1}{2} \int \rho |\delta \vec{u}|^2 dV \approx \frac{l(l+1)}{2} \omega_r^2 \int_0^R \rho a^4 |T_{ll}|^2 da \quad (58)$$

is the energy of an *r*-mode measured in the rotating frame. Using (46) we can deduce that (for an $n = 1$ polytrope)

$$E \approx 10^{51} \alpha^2 M_{1.4} R_{10}^2 P_{-3}^{-2} \text{ erg} \quad (59)$$

We focus our discussion on the $l = m$ modes because they have been shown to lead to the strongest instability^{38,60}. Note also that we only keep the leading order terms in (58) and the dissipation integrals below. For simplicity, we also use the Cowling approximation, an assumption that only introduces an inaccuracy of a few percent^{29,30}, cf. Table 1. This is rather good precision in

an astrophysical context, and there are far larger uncertainties in our current understanding of the r -mode instability.

Ideally one would like to address issues regarding the relevance of the supranuclear equation of state for an active mode-instability. However, to use a realistic equation of state in a Newtonian calculation is not particularly meaningful. The reason for this is that there is no one-to-one correspondence between Newtonian and relativistic stellar models in the sense that, for a given equation of state, the same central density leads to stars with comparable masses and radii in the two theories. Thus we believe that a calculation for realistic equations of state in the Newtonian framework can be misleading. Still, it is interesting to try to understand whether the overall properties of the equation of state, eg. the stiffness, affect the instability. This issue can be studied by considering different polytropic models, and in the following we consider results both for $n = 1$ polytropes and constant density stars (represented by the limit $n \rightarrow 0$). These results ought to be indicative for realistic neutron stars given that the average adiabatic index of most realistic equations of state lies in the range $n = 0 - 1$.

4.1. *Gravitational radiation reaction*

The first step in establishing the relevance of the r -mode instability corresponds to showing that the modes grow on an astrophysically interesting timescale. To estimate the gravitational-radiation reaction we use the standard post-Newtonian multipole-formulas (see Thorne²⁷). The gravitational-wave luminosity associated with a pulsation mode (measured in the rotating frame) is then estimated using

$$\frac{dE}{dt} \Big|_{\text{gw}} = -\omega_r \sum_{l=2}^{\infty} N_l \omega_i^{2l+1} (|\delta D_{lm}|^2 + |\delta J_{lm}|^2) , \quad (60)$$

where

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2} . \quad (61)$$

The first term in the bracket of (60) represents radiation due to the mass multipoles. These are determined by

$$\delta D_{lm} = \int \delta \rho a^l Y_{lm}^* dV . \quad (62)$$

The second term in the bracket of (60) corresponds to the current multipoles, which follow from

$$\delta J_{lm} = \frac{2}{c} \sqrt{\frac{l}{l+1}} \int a^l (\rho \delta \vec{u} + \delta \rho \vec{\Omega}) \vec{Y}_{lm}^{B*} dV , \quad (63)$$

where \vec{Y}_{lm}^B are the magnetic multipoles²⁷.

Table 2. Estimated growth times due to current and mass-multipole gravitational radiation. The results are for the single $l = m$ r-mode that exists for isentropic stars, and are given both for constant density stars ($n = 0$) and $n = 1$ polytropes. We do not show mass multipole results for the $\rho = \text{constant}$ case, since $\delta\rho = 0$ in that case.

| l | current multipole (t_{cm} in s) | | mass multipole (t_{mm} in s) |
|-----|------------------------------------|-------------------|---------------------------------|
| | $n = 0$ | $n = 1$ | $n = 1$ |
| 2 | 22 | 47 | 3.6×10^5 |
| 3 | 4.7×10^2 | 1.3×10^3 | 4.3×10^6 |
| 4 | 1.0×10^4 | 3.2×10^4 | 8.5×10^7 |

An order count based on the above formulas suggests an interesting result: For $l = m$ r-modes, the dominant contribution to the gravitational radiation comes from the first term in (63). That this is the case can be seen as follows. Recall that the $l = m$ modes have toroidal displacement to leading order, while the spheroidal perturbations enter at order Ω^2 . Furthermore, we know that if the toroidal component corresponds to the l th multipole, the spheroidal components will correspond to $l + 1$. This means that, as a gravitational-wave source the r -modes are quite unusual. Since they are primarily perturbations of the velocity field in the star, with little disturbance in the star's density, the gravitational radiation that they emit comes primarily from the time-dependent mass currents (we see that $\delta D_{lm} \sim \Omega^2$ while $\delta J_{lm} \sim \Omega$). This is the gravitational analogue of magnetic multipole radiation. In fact, the r -mode instability is unique among expected astrophysical sources of gravitational radiation in radiating primarily by gravitomagnetic effects.

After inserting the leading order r -mode eigenfunction in the relevant current multipole term, we find

$$\frac{dE}{dt} \bigg|_{\text{gw}}^{\text{current}} \approx -4l^2 N_l \omega_r^3 \omega_i^{2l+1} \left| \int_0^R \rho a^{l+3} T_{ll} da \right|^2 \quad (64)$$

for the rate of energy loss due to gravitational waves. This then leads to an estimated growth timescale for the instability

$$t_{\text{gw}}^{\text{current}} \approx -t_{cm} M_{1.4}^{-1} R_{10}^{-2l} P_{-3}^{2l+2} \text{ s} \quad (65)$$

where the negative sign indicates that the mode is unstable.

The estimated current multipole timescales are given (for $l = m$ r-modes of an isentropic star) in Table 2. The listed results are consistent with original ones obtained by several authors^{38,39,60,40}. From the tabulated results we can deduce two things. First of all we see that the timescale increases by roughly one order of magnitude with each l . Thus, higher multipoles lead to significantly weaker instabilities and the $l = m = 2$ r-mode will be the most important. Secondly, we can see that the results for constant density stars and $n = 1$ polytropes differ by roughly a factor of two. This provides a useful illustration of the uncertainties associated with the supranuclear equation of state.

It is worth pointing out that the conclusion that the current multipoles dominate the gravitational-wave emission is not necessarily true for all the r -modes. It is certainly true for isentropic stars, since such stars have no “pure” r -modes for $l \neq m$. But in the general non-isentropic case we have to consider the infinite set of modes that exist for all $m \neq 0$. An order count in (60) immediately shows that the situation is not so simple for the $l \neq m$ modes. In general, the mode-solution for these modes have contributions to the toroidal eigenfunction described by the $[l, m]$ multipole, but the spheroidal part and the density perturbation (proportional to Ω^2) correspond to $[l \pm 1, m]$. Consequently, the current multipoles and the mass multipoles contribute to (60) at the same order in Ω for many of the non-isentropic modes. Thus mass multipole radiation could play an important role. It is also relevant to stress that the mass multipoles are far from irrelevant for the isentropic $l = m$ modes. From (62) we find that the leading contribution to the mass multipole radiation follows from

$$\frac{dE}{dt} \bigg|_{\text{gw}}^{\text{mass}} \approx -4l^2 N_l \omega_r^3 \omega_i^{2l+1} \left| \int_0^R \frac{\rho^2 g a^{l+3}}{\Gamma p} \zeta_{l+1} da \right|^2, \quad (66)$$

where we have introduced

$$\delta p_{lm} = \rho g a \zeta_{lm}. \quad (67)$$

This leads to

$$t_{\text{gw}}^{\text{mass}} \approx -t_{mm} M_{1.4} R_{10}^{-2l-6} P_{-3}^{2l+6} \text{ s} \quad (68)$$

where the estimated values for t_{mm} are listed in Table 2. Here it is interesting to note that the mass and current multipole results scale with the size of the star in rather different ways. This means that the relative importance of the two is significantly different for stars of different size.

4.2. Viscous damping

The estimates given in Table 2 show that the r -modes grow rapidly enough to be of potential significance. Still, they must also overcome damping effects due to different physical mechanisms, most of which are poorly understood and therefore difficult to model in a realistic fashion.

As in the case of the unstable f -modes the main dissipation in a hot newly born neutron star is due to viscosity. For the r -modes to be relevant they must grow fast enough that they are not completely damped out by viscosity. To assess the strength of the viscous damping of the r -modes we implement the approximations that have been used to study unstable f -modes in the Newtonian context⁵⁴. For the simplest neutron star models two kinds of viscosity, bulk and shear viscosity, are normally considered. These are due to rather different physical mechanisms and we discuss them separately below.

4.2.1. Shear viscosity

At relatively low temperatures (below a few times 10^9 K) the main viscous dissipation mechanism in a fluid star arises from momentum transport due to particle scattering. In the standard approach these scattering events are modelled in terms of a macroscopic shear viscosity. In a normal fluid star neutron-neutron scattering provides the most important contribution. In a superfluid the situation is a bit more subtle. Electron-electron scattering leads to the dominant contribution to the shear viscosity, but one must also account for exotic effects like the scattering off of vortices in the superfluid, cf. Section 4.5.

The effect of shear viscosity on the *r*-modes can be estimated from^{60,38}

$$\frac{dE}{dt} \Big|_{sv} = -2 \int \eta \delta \sigma^{ab} \delta \sigma_{ab}^* dV \quad (69)$$

where the shear σ_{ab} follows from

$$\delta \sigma_{ab} = \frac{i\omega_r}{2} (\nabla_a \xi_b + \nabla_b \xi_a - 2g_{ab} \nabla_c \xi^c) \quad (70)$$

After working out the angular integrals we are left with

$$\frac{dE}{dt} \Big|_{sv} = -\omega_r^2 l(l+1) \left\{ \int_0^R \eta a^2 |a \partial_a T_{ll}|^2 da + (l-1)(l+2) \int_0^R \eta a^2 |T_{ll}|^2 da \right\} . \quad (71)$$

Above the transition temperature at which the neutron star becomes superfluid (several times 10^9 K), the appropriate viscosity coefficient (due to neutron-neutron scattering) is

$$\eta_{nn} = 2 \times 10^{18} \rho_{15}^{9/4} T_9^{-2} \text{g/cms} . \quad (72)$$

where

$$\rho_{15} = \rho / 10^{15} \text{g/cm}^3 \quad (73)$$

$$T_9 = T / 10^9 \text{K} \quad (74)$$

This leads to an estimated dissipation time-scale due to shear viscosity^{38,60,39}

$$t_{sv} \approx M_{1.4}^{-5/4} R_{10}^{23/4} T_9^2 \times \begin{cases} 1.2 \times 10^8 \text{s} & n = 0 \\ 6.7 \times 10^7 \text{s} & n = 1 \end{cases} \quad (75)$$

To obtain these results the star has been assumed to be isothermal. This should be a reasonable approximation apart from for the first few moments of a neutron stars life. These estimates of the shear viscosity are expected to be relevant for the first months of the life of a hot young neutron star.

Once the star has cooled below the superfluid transition temperature, the above viscosity coefficient must be replaced by

$$\eta_{ee} = 6 \times 10^{18} \rho_{15}^2 T_9^{-2} \text{g/cms} . \quad (76)$$

which follows from an analysis of electron-electron scattering. This leads to^{38,60,39}

$$t_{sv} \approx M_{1.4}^{-1} R_{10}^5 T_9^2 \times \begin{cases} 3.6 \times 10^7 \text{ s} & n = 0 \\ 2.2 \times 10^7 \text{ s} & n = 1 \end{cases} \quad (77)$$

The above shear viscosity estimates were based on standard Navier-Stokes theory as developed by, for example, Landau and Lifshitz⁶¹. However, this theory is somewhat pathological in that it allows dissipative signals to travel faster than light. A causal theory of dissipation has been developed following initial work by Israel and Stewart⁶². This alternative description differs from Navier-Stokes theory in some potentially important ways. The main feature of the Israel-Stewart description is that there is a coupling between rotation and the gradients of both temperature and momentum. Intuitively one might expect these additional effects to be small, but there are two situations where they may be relevant. For high frequency oscillations the large time-derivatives may lead to significant corrections to the Navier-Stokes results. Similarly, rapid rotation can lead to a strong coupling. The latter effect could certainly be relevant for the r -modes (or indeed any other mode in a rapidly spinning star). Inspired by this possibility, Rezania and Maartens⁶³ have estimated the relevance of the coupling between vorticity and shear viscosity for the r -modes. Their results show that the correction due to the vorticity coupling can be large, especially for low temperatures.

4.2.2. Bulk viscosity

At high temperature (above a few times 10^9 K) bulk viscosity is the dominant dissipation mechanism. Bulk viscosity arises because the pressure and density variations associated with the mode oscillation drive the fluid away from beta equilibrium. It corresponds to an estimate of the extent to which energy is dissipated from the fluid motion as the weak interaction tries to re-establish equilibrium. The mode energy lost through bulk viscosity is carried away by neutrinos.

In the previous section we found that we could calculate the shear viscosity timescale from the leading order contribution to the r -mode fluid motion, cf. (71). This is not the case for the bulk viscosity. To assess its relevance we need the Lagrangian density/pressure perturbation. But for r -modes these perturbations vanish to leading order in rotation. This means that the mode calculation must be carried at least to order Ω^2 if we want to estimate the bulk viscosity timescale. In particular, we need

$$\frac{dE}{dt} \Big|_{bv} = - \int \zeta |\delta\sigma|^2 \quad (78)$$

where $\delta\sigma$ is the expansion associated with the mode, defined by

$$\delta\sigma = -i\omega_r \frac{\Delta\rho}{\rho} = -i\omega_r \frac{\Delta p}{\Gamma p} \quad (79)$$

and

$$\Delta p = \delta p + \frac{dp}{da} \xi^a = \rho g a \sum_{lm} [\zeta_{lm} - S_{lm}] Y_{lm} , \quad (80)$$

where we have used the fact that $dp/da = -\rho g$ for a Newtonian model. The detailed calculation of the relevant eigenfunctions has been described by, for example, Saio³⁰.

After doing the angular integrals we get (for $l = m$)

$$\frac{dE}{dt} \Big|_{\text{bv}} = - \int_0^R \zeta \left(\frac{\rho g a^2}{\Gamma p} \right)^2 [\zeta_{l+1l} - S_{l+1l}]^2 da . \quad (81)$$

In the case when β -equilibrium is regulated by the so-called modified URCA reactions the relevant viscosity coefficient is

$$\zeta_{mU} = 6 \times 10^{25} \rho_{15}^2 T_9^6 \left(\frac{\omega_r}{1\text{Hz}} \right)^{-2} \text{g/cms} . \quad (82)$$

This leads to an estimated bulk viscosity timescale for $n = 1$ polytropes³⁷

$$t_{\text{bv}} \approx 2.7 \times 10^{11} M_{1.4} R_{10}^{-1} P_{-3}^2 T_9^{-6} \text{ s} \quad (83)$$

This result, which was obtained within the Cowling approximation agrees well with a calculation including the perturbed gravitational potential⁴⁰.

We note that it has been speculated that the proton fraction may be large enough to make direct URCA reactions possible in the cores of neutron stars. If this is the case, the bulk viscosity coefficient becomes significantly larger than (82). However, direct URCA processes will only be relevant if the proton fraction is above several percent^{64,65}. Since this is unlikely to be the case in the surface regions where the r -modes are mainly located we will not consider the direct URCA bulk viscosity in detail here.

Let us also point out that, since very hot stars are no longer transparent to neutrinos, the above estimates for the bulk viscosity will not remain relevant at very high temperatures⁶⁶. This may be of some relevance for the very early phase of r -mode growth in a newly born neutron star.

Another point, that may turn out to be of crucial importance, was recently raised by Jones⁶⁷. He pointed out that the bulk viscosity result would be significantly different if the presence of hyperons in the neutron star core was accounted for. Not only is the associated viscosity coefficient stronger than assumed above, the temperature dependence is also different in the hyperon case (the coefficient scales as T^{-2} rather than T^6). This makes the hyperon bulk viscosity relevant at low temperatures. Jones argues that the r -mode instability is almost completely suppressed in the case when hyperons are present throughout the entire star. This is, however, not a particularly realistic assumption. In order to assess the extent to which hyperons affect the estimated instability

time-scales calculations that allow for the presence of exotic particles (as predicted by modern equation of state results) need to be performed. At the time of writing, no such results are available.

Since there has been some debate in the literature regarding the estimated timescales for the bulk viscosity it is relevant that we try to identify the reasons why various given estimates differ. First of all, let us discuss the constant density estimate of Kokkotas and Stergioulas³⁹. As is easy to see from (79), one cannot rigorously calculate the expansion associated with a mode in a constant density star. After all, the assumption $\rho = \text{constant}$ immediately leads to $\Delta\rho = 0$. Still, one can estimate of the relevant effect (following Cutler and Lindblom⁷⁰) in the following way: Assume that the Lagrangian variation in the pressure takes the following form

$$\frac{\Delta p}{p} \approx \frac{\epsilon l}{\Gamma} \left(\frac{a}{R}\right)^l e^{i\omega_r t} Y_{lm} \quad (84)$$

where ϵ is a dimensionless normalisation constant. This formula then leads to the anticipated behaviour as $\Gamma \rightarrow \infty$, i.e. as $\rho \rightarrow \text{constant}$, and the formula also agrees with the result for f -mode oscillations in compressible stars (to within a factor of two). Kokkotas and Stergioulas³⁹ use this approximation also for the r -modes, and arrive at an estimated bulk viscosity timescale (actually using $\Gamma = 5$ in (84))

$$t_{\text{bv}} \approx 2.4 \times 10^{10} M_{1.4}^{-1} R_{10}^5 T_9^{-6} P_{-3}^2 \text{ s} \quad (85)$$

Compared to (83) this is an underestimate of the strength of bulk viscosity dissipation by roughly one order of magnitude.

An estimate that was equally far away from the correct value was given by Lindblom, Owen and Morsink⁶⁰. They tried to approximate the bulk viscosity using only first order (in Ω) results. In particular, they assumed that $\Delta\rho \approx \delta\rho$. This assumption is, however, not warranted for the r -modes. As we discussed in Section 2.3, one of the fundamental r -mode properties is that the fluid essentially moves on isobars in the perturbed configuration. Hence, we will have $|\Delta p| \ll |\delta p|$, which results in Lindblom, Owen and Morsink overestimating the strength of the bulk viscosity by roughly one order of magnitude.

Finally, the first published second order (in Ω) results (due to Andersson, Kokkotas and Schutz³⁸) also differ considerably from (83), essentially in the scaling with the rotation rate. This was due to a typographical error in the numerical code used to calculate the mode eigenfunctions. Once this mistake is corrected, we arrive at (83).

4.3. The r -mode instability window

Armed with the estimates obtained in the previous two sections we can address the main issue of interest: Will the r -mode instability be relevant for astrophysical neutron stars? First of all, it is easy to see that the r -modes will only be unstable in a certain range of temperatures. To have an instability we

need t_{gw} to be smaller in magnitude than both t_{sv} and t_{bv} . From the various estimates we immediately see that the dissipation due to shear viscosity kills the mode at low temperatures, while the bulk viscosity dominates at high temperatures. In fact, by comparing (77) to (65) we deduce that shear viscosity will completely suppress the *r*-mode instability at core temperatures below 10^5 K. Similarly, bulk viscosity will prevent the mode from growing in a star that is hotter than a few times 10^{10} K (but see the proviso in section 4.2.2 regarding the relevance of the bulk viscosity at high temperatures). In the intermediate range there is a window of opportunity where the growth time due to gravitational radiation is short enough to overcome the viscous damping and drive the *r*-mode unstable.

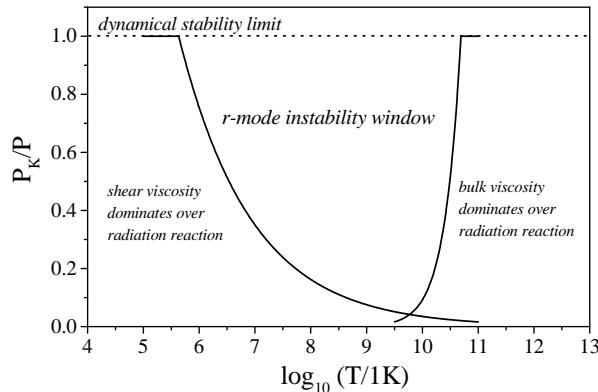


Fig. 3. The critical rotation rates at which shear viscosity (at low temperatures) and bulk viscosity (at high temperatures) balance gravitational radiation reaction due to the *r*-mode current multipole. This leads to the notion of a “window” in which the *r*-mode instability is active. The data in the figure is for the $l = m = 2$ *r*-mode of a canonical neutron star ($R = 10$ km and $M = 1.4M_{\odot}$ and Kepler period $P_K \approx 0.8$ ms).

This instability window is usually illustrated by the critical rotation period, above which the mode is unstable, as a function of temperature^{54,60,38}. We find the relevant critical rotation rate by solving for the zeros of

$$\frac{1}{2E} \frac{dE}{dt} = \frac{1}{t_{\text{gw}}} + \sum \frac{1}{t_{\text{diss}}} = 0 \quad (86)$$

for a range of temperatures. A typical result (for canonical neutron star parameters) is shown in Figure 3. For canonical values, the Kepler limit corresponds to $P_K = 0.8$ ms. In the figure we show the critical period curve for an $n = 1$ polytrope. From this data we can deduce that a rotating neutron star with a core temperature of 10^9 K (i.e. a few months old) is unstable at rotation periods shorter than 25 ms. There is, of course, a large uncertainty associated with this

prediction. If we, for example, assume that the instability sets in at roughly the same P_K/P for different equations of state (which is supported by the $n = 0$ and 1 results), then the uncertainty in the Kepler period ($P_K \approx 0.5 - 2$ ms) for realistic equations of state suggests that the instability will be relevant at rotation rates faster than $P \approx 10 - 40$ ms at some core temperature.

Figure 5 shows the different strengths of the normal fluid shear viscosity (due to neutron-neutron scattering) and the superfluid counterpart (due to electron-electron scattering). It is interesting to note that the superfluid is, rather counterintuitively, more dissipative than the normal fluid. We also indicate the strength of the $l = m = 2$ *r*-mode mass multipole. As a comparison we should recall that the $m = 4$ *f*-mode (believed to be the “most unstable” spheroidal mode) becomes unstable above $P_K/P \approx 0.95$. In other words, the *r*-mode instability provides a much stronger constraint on neutron star rotation than the instability in the *f*-mode does. In fact, even the comparatively weak radiation reaction associated with the *r*-mode mass multipoles leads to an instability that completely dominates the *f*-mode one.

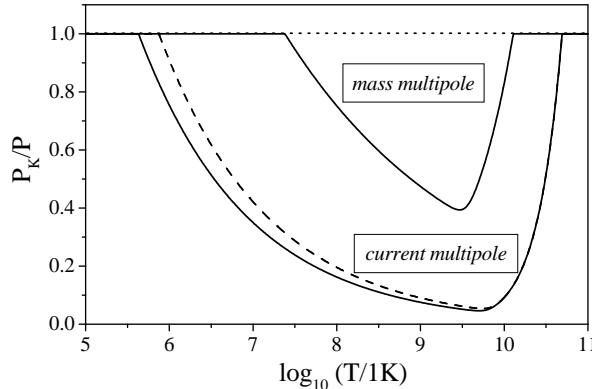


Fig. 4. The *r*-mode instability window for a canonical neutron star (the data is for an $n = 1$ polytrope). The lower two curves illustrate the difference between shear viscosity due to neutron-neutron scattering (solid line) and electron-electron scattering (dashed curve). The latter should be relevant for a superfluid star (i.e. below a few times 10^9 K). It should be noted that the superfluid shear viscosity is stronger than that of a normal fluid. We also show the instability window that arises if we only include the mass multipole radiation from the $l = m = 2$ *r*-mode. It is useful to compare these results to those for the $m = 4$ *f*-mode, that suggest that this mode is unstable for $P_K/P > 0.95$ or so.

Finally, we illustrate the relative importance of the higher $l = m$ *r*-modes in Figure 5. It should be noticed that, even though the growth time increases with roughly one order of magnitude for each l the higher multipole *r*-modes still provide a severe constraint on neutron star rotation rates.

The results in Figures 3-5 are encouraging and perhaps indicative of the

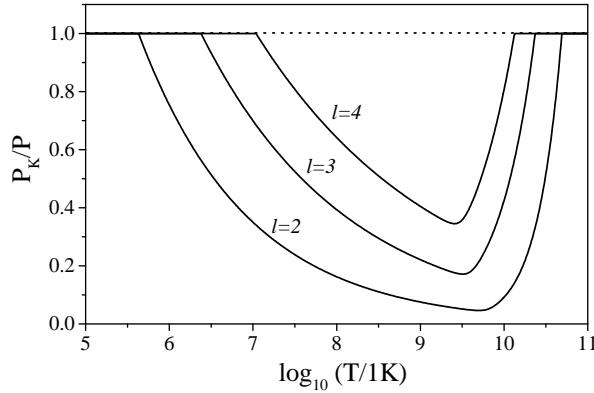


Fig. 5. The *r*-mode instability window for a canonical neutron star (the data is for an $n = 1$ polytrope) and different l -multipoles.

astrophysical role of the *r*-mode instability[†]. But more than anything else they emphasize the urgent need for an improved understanding of the various elements in our model. We will now discuss various attempts to improve on the detailed physics included in the *r*-mode calculations.

4.4. Superfluid mutual friction

As a neutron star cools below a few times 10^9 K the extreme density in the core is expected to lead to the formation of a superfluid^{71,72}. This means that, while it may not be wholly inappropriate to model a newly born neutron star as a simple “ball” of a more or less perfect fluid, a model of an old neutron star must certainly be more complicated. In many ways it is useful to think of an old (which in this context means: older than a few hours/days) neutron star as a “layer cake”. A simple, yet reasonable, picture of an old neutron star consists of i) a core in which superfluid neutrons (in a 3P_2 condensate) coexist with superconducting protons (above $\rho \approx 1.5 \times 10^{14}$ g/cm³), ii) a crust: a lattice of nuclei permeated by superfluid neutrons in a 1S_0 state (the superfluid reaches out to the neutron drip density $\sim 10^{11}$ g/cm³), and iii) a fluid ocean. A canonical neutron star would have a core of 9 km, a crust of 1 km and a thin ocean 50-100 meters deep. The superfluid constituents play a crucial role in determining the dynamical properties of a rotating neutron star. In particular, the interplay between the lattice nuclei and the superfluid in the inner crust is a key agent in the standard model for neutron star glitches. Intuitively, the presence of a superfluid should have a considerable effect also on the various

[†]We note in passing that the *r*-modes may also be unstable in the fastest rotating white dwarfs^{109,68,69}; the so-called DQ Herculis stars.

modes of pulsation. To some extent this is already indicated by the results in Figure 5, and when the presence of new dissipation mechanisms is accounted for it turns out that a superfluid may be more dissipative than a normal fluid! The most important new effect is the so-called mutual friction that arises from scattering of electrons off of the neutron vortices (recall that a superfluid mimics large scale rotation by forming a large number of vortices). This scattering is greatly enhanced by the strong magnetic field (due to the entrained protons) within the vortices.

The motion associated with a neutron star pulsation mode is of large scale compared to both the neutron vortex and superconducting proton magnetic flux tube separations ($\sim 10^{-3}$ cm and $\sim 10^{-10}$ cm, respectively). This means that it is sufficient to include the averaged dynamics of many vortices and flux tubes in a model for superfluid stellar pulsation. This leads to models consisting of two distinct, but dynamically coupled, fluids representing the superfluid neutrons and the “protons” (a fluid that contains all charged components in the star). The dynamics can then be described in terms of two velocity fields, described by two sets of equations similar to (12) and (13) with additional coupling terms (see equations (1) to (4) in Lindblom and Mendell⁷³). It is interesting to note that this superfluid model leads to a new set of pulsation modes already in the nonrotating case^{74,75}. In order to discuss the difference between these modes and the familiar f/p -modes it is useful to introduce a variable $\delta\beta$, which represents the “departure from β -equilibrium” in the oscillating fluid. Then the standard fluid modes all have small $\delta\beta$. For these modes the superfluid neutrons and the protons move more or less together. In contrast, the superfluid modes have $\delta\beta$ large, and the neutrons and protons are largely countermoving.

Superfluid mutual friction has been shown to completely suppress the instability associated with the f -mode in a rotating star⁷⁶. Plausibly, it is a dominant damping agent also on the unstable r -modes. We can assess the relevance of mutual friction in a hand-waving way following Mendell⁷⁷: For a longitudinal wave travelling perpendicular to the rotation axis the influence of mutual friction can be estimated as

$$t_{\text{mf}} \approx \frac{\Gamma_{\text{mf}}}{\Omega} \frac{1}{\lambda^2} t_{\text{sv}} \quad (87)$$

Here λ is the wavelength of the oscillation (which we will take to be $\lambda \approx R \approx 10$ km), and

$$\Gamma_{\text{mf}} < 10^{10} T_9^{-2} \text{ cm}^2/\text{s} \quad (88)$$

Comparing this to our estimated shear viscosity timescale we see that

$$t_{\text{mf}} > 200 \text{ s} \quad (89)$$

and that this timescale is independent of the temperature. This is in clear contrast with the shear and bulk viscosities, both of which are sensitive to changes in the core temperature. This result suggests that, even though mutual

friction may be important it will not completely suppress the *r*-mode instability. In fact, we would expect a detailed calculation to lead to timescales considerably longer than this upper limit. This is since mutual friction would have the largest effect on waves that travel perpendicular to the rotation axis (as in our example) while the velocity field of a pulsation mode has a considerable component parallel to the axis.

In the “macroscopic” picture, the strength of mutual friction can be expressed in terms of the “entrainment parameter”

$$\epsilon = \frac{\rho_p}{\rho_n} \left(\frac{m_p^*}{m_p} - 1 \right) \quad (90)$$

where ρ_n and ρ_p are the neutron and proton densities, respectively, and m_p and m_p^* are the bare and effective proton masses. Estimates of the ratio of these two masses suggest that⁷⁸

$$0.3 \leq \frac{m_p^*}{m_p} \leq 0.8 \quad (91)$$

Steps towards more detailed studies of superfluid *r*-modes have been taken by Lindblom and Mendell⁷⁸. They have calculated the modes in a superfluid star and shown that they are identical to those of a normal fluid star to lowest order in Ω . Therefore the growth timescale due to gravitational radiation reaction remains essentially unchanged in a superfluid star, cf. (65). For the specific two-fluid stellar model considered by Lindblom and Mendell, the shear viscosity timescale turns out to be roughly a factor of 3 shorter than in the normal fluid case, cf. (75). As far as the mutual friction is concerned, the calculations of Lindblom and Mendell do not provide a complete answer to the question of whether this mechanism will suppress the *r*-mode instability or not.

Lindblom and Mendell find that the typical result for *r*-mode dissipation due to mutual friction is⁷⁸

$$t_{\text{mf}} \approx 2 \times 10^5 P_{-3}^5 \text{ s} . \quad (92)$$

This means that the *r*-mode instability window in (say) Figure 3 would be essentially unaffected by the inclusion of mutual friction. However, as manifested in Figure 6 of Lindblom and Mendell, the result is sensitive to changes in the entrainment parameter. In particular, there are critical values of ϵ for which the mutual friction timescale becomes very short (and for which the unstable *r*-mode would be completely suppressed). It is interesting to note that $\delta\beta$ is a key parameter in deciding the importance of mutual friction. The critical values of ϵ for which the mutual friction timescales are short correspond to modes for which $\delta\beta$ is comparatively large. As already mentioned, the protons and neutrons are essentially countermoving for large $\delta\beta$. Given that mutual friction tends to damp out the relative motion between the neutrons and the protons, it is natural that modes with large $\delta\beta$ should be strongly affected by it. The

results of Lindblom and Mendell seem to suggest that there are values of the entrainment parameter for which the r -modes are in “resonance” with a superfluid mode (with large $\delta\beta$). This would explain why these parameter values lead to significantly increased dissipation. Further studies of this problem are clearly needed if we are to understand the r -modes in a superfluid star fully. Investigations of this problem should be strongly encouraged, both in Newtonian theory and general relativity⁷⁹.

4.5. *Ekman layer in neutron stars with a crust*

In addition to the core superfluid, we need to consider effects due to the presence of a solid crust in an old neutron star. The melting temperature of the crust is usually estimated to be of the order of 10^{10} K (for a non-accreting neutron star), so the crust may form shortly after the neutron stars birth⁸⁰.

That the presence of a solid crust will have a crucial effect on the r -mode motion can be understood as follows: Based on the perfect fluid mode-calculations we anticipate the transverse motion associated with the mode at the crust-core boundary to be large. However, if the crust is assumed to be rigid the fluid motion must essentially fall off to zero at the base of the crust in order to satisfy a no-slip condition (in the rotating frame of reference). We can estimate the relevance of the crust using viscous boundary layer theory⁸¹. The region immediately beneath the crust then corresponds to a so-called Ekman layer. The thickness of the boundary layer (δ) can be deduced by balancing the Coriolis force and shear viscosity⁴¹:

$$\delta \sim \left(\frac{\eta}{\rho\Omega} \right)^{1/2} \quad (93)$$

where η is the shear viscosity coefficient (72). As is easy to show, this would correspond to a few centimetres for typical parameters of a rapidly rotating neutron star. In other words, $\delta \ll R$ and the boundary layer approximation is warranted.

The dissipation timescale due to the presence of the Ekman layer roughly follows from

$$t_{\text{Ek}} \approx \frac{t_{\text{sv}}}{\sqrt{Re}} \quad (94)$$

where $Re = \rho_b R_b^2 \Omega / \eta$ is the Reynolds number (the ratio between the Coriolis force and viscosity), and R_b and ρ_b are the location of, and density in, the Ekman layer, respectively. For typical neutron star parameters one would expect the base of the crust to lie at $\rho_b \approx 1.5 \times 10^{14}$ g/cm³. We would then have $Re \sim 10^{13}$, and we see that the presence of a solid crust leads to a dissipation channel that is many orders of magnitude more effective than the standard shear viscosity.

A more detailed estimate of this effect has been made by Bildsten and Ushomirsky⁸¹. The required dissipation rate follows from an integral over the

surface area at the crust-core boundary

$$\frac{dE}{dt} \Big|_{Ek} = -R_b^2 \int \frac{|\delta \vec{u}|^2}{2} \left(\frac{\rho_b \omega_r \eta}{2} \right)^{1/2} \sin \theta d\theta d\varphi \quad (95)$$

For canonical neutron star parameters and an $n = 1$ polytrope we can estimate that the base of the crust corresponds to $R_b \approx 9.4$ km. When combined with the estimated mode-energy, this leads to an estimated damping timescale (note that this estimate is valid only for a $R = 10$ km and $M = 1.4M_\odot$ star)

$$t_{Ek} \approx 1.4 \times 10^3 T_9 P_{-3}^{1/2} \text{ s} \quad (96)$$

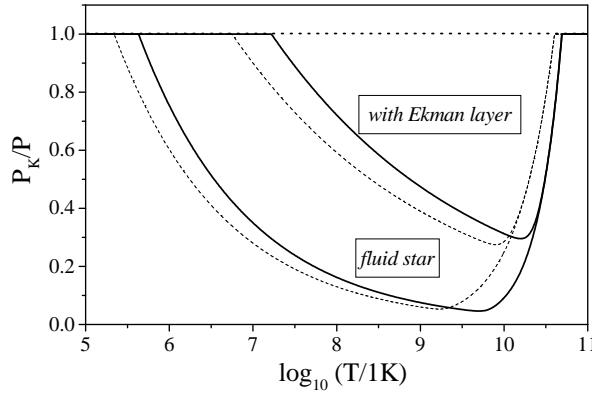


Fig. 6. The r -mode instability window for a canonical fluid neutron star (lower solid curve) is compared to that for a star with an Ekman layer operating at the base of the solid crust (upper solid curve). Also shown (as dashed curves) are the corresponding instability windows for a constant density star. It should be noted that the strength of the r -mode instability is only weakly dependent on the stiffness of the equation of state.

This estimate was arrived at by appealing to an analogy between the problem at hand and the standard one of an oscillating plate in a viscous fluid⁶¹. One would not expect the change in geometry and inclusion of the Coriolis force in the r -mode problem to change the order of magnitude of the effect. That this assumption is correct has been shown by Rieutord⁸², who used an exact solution to the incompressible problem to show that when the angular dependence of the velocity in the Ekman layer is accounted for the dissipation timescale for the $l = m = 2$ r -mode is a factor of 1.74 shorter than (96). The resultant estimate of the dissipation due to the presence of the Ekman layer is

$$t_{Ek} \approx 830 T_9 P_{-3}^{1/2} \text{ s} . \quad (97)$$

We compare the r -mode instability window obtained from this estimate to the normal fluid result in Figure 6. Clearly, a solid crust has an influence that is

far greater than any previously considered dissipation mechanism. For example, our estimate suggests that all neutron stars with a crust are stable at rotation periods longer than roughly 5 ms.

The crust-core interface has been the focus of several recent studies. These studies add further dimensions to the problem. The interplay between the r -modes in the fluid core and modes in the solid crust is particularly interesting. It has long been known that the crust supports toroidal modes of oscillation (whose frequencies depend directly on the crust shear modulus)^{17,18}. Detailed calculations by Yoshida and Lee⁸⁴ (see also Levin and Ushomirsky⁸⁵) show that as the spin of the star increases the unstable r -modes will undergo a series of so-called avoided crossings with the crust modes. In a nonrotating star, the fundamental $l = 2$ crust mode has a frequency

$$\omega_c(\Omega = 0) \approx 10^{-2} \sqrt{\frac{GM}{R}} . \quad (98)$$

In the rotating case the mode-frequency changes in such a way that

$$\omega_c(\Omega) \approx \omega_c(\Omega = 0) + \frac{m\Omega}{l(l+1)} \quad (99)$$

(in the rotating frame). Now it is easy to see that the toroidal mode becomes comparable to the r -mode frequency at a rotation frequency^{18,83}

$$\Omega_{\text{cross}} \approx 5 \times 10^{-2} \Omega_K \quad (100)$$

Hence, avoided crossings between the modes occur at very low rates of rotation. This is conceptually interesting and it may have repercussions on many of our estimates regarding the r -mode instability. Most importantly, it casts doubt on extrapolations of the slow-rotation results into the regime of fast rotation.

In addition to studying this new feature of the problem, Levin and Ushomirsky⁸⁵ show that the assumption of a rigid crust, which was made in the above estimates of the Ekman layer dissipation rate, is likely not warranted. They show that the r -mode typically extends into the crust. We can quantify the extent to which this affects the estimated dissipation timescale (97) in terms of the ratio $\Delta = |\Delta v|/|v|$, where Δv is the difference between the core fluid velocity immediately beneath the crust and the velocity induced in the crust and v is the fluid velocity in absence of the crust. The rigid crust assumption corresponds to $\Delta = 1$. In contrast, the toy model calculations of Levin and Ushomirsky⁸⁵ suggest that a typical results should be $\Delta \approx 0.1$. This then affects the Ekman layer timescale by a factor of $1/\Delta^2$, so the true dissipation may well be a factor of at least 100 weaker than (97).

Despite some recent advances in our understanding of the effects that a solid crust may have on the r -modes it is clear that several crucial issues remain to be investigated in detail. For example, the inner crust of a neutron star (out to

the neutron drip density) will likely be permeated by superfluid neutrons. It is not at all clear at the moment whether one should expect these neutrons to be strongly pinned to the crust nuclei or not. But if the superfluid is at all free to move relative to the crust it will likely lead to a weaker Ekman layer dissipation on the *r*-modes.

4.6. Strange/quark stars

Ever since Witten⁸⁶ suggested that strange matter (a conglomerate of strange, up and down quarks) would be the most stable form of matter, it has been speculated that strange stars might exist in the universe^{87,88,89}. It has been suggested that some (perhaps all) pulsars are actually strange stars. The observational evidence for this is, however, rather tenuous. The main reason for this is that strange stars are held together by both the strong interaction and gravity, and at a mass of one and a half solar masses, gravity dominates over the strong interaction. Thus it is not easy to distinguish a strange star from a neutron star, particularly not since the strange star is expected to be “covered” by a thin nuclear crust⁸⁷. Still, the existence of strange stars remains a conjecture which is consistent with, for example, millisecond pulsar observations.

As was pointed out by Madsen⁹⁰, the *r*-mode instability may provide the means for distinguishing between strange stars and neutron stars. The main reason for this is that the viscosity coefficients are rather different in the two cases. While the shear viscosity in a strange star would be comparable to that of a neutron star, the bulk viscosity would be many orders of magnitude stronger than its neutron star counterpart. This has interesting effects on the *r*-mode instability.

For the shear viscosity the relevant coefficient would be

$$\eta \approx 1.7 \times 10^{18} \left(\frac{0.1}{\alpha_s} \right)^{5/3} \rho_{15}^{14/9} T_9^{-5/3} \text{ g/cms} \quad (101)$$

where α_s is the fine-structure constant for the strong interaction. This leads to

$$t_{sv} \approx 7.4 \times 10^7 \left(\frac{\alpha_s}{0.1} \right)^{5/3} M_{1.4}^{-5/9} R_{10}^{11/3} T_9^{5/3} \text{ s} \quad (102)$$

Meanwhile, the situation is more complicated for the bulk viscosity (which now is a result of the change in concentration of down and strange quarks in response to the mode oscillation). The relevant viscosity coefficient takes the form^{90,91}

$$\zeta = \frac{\alpha T^2}{\omega_r^2 + \beta T^4} \quad (103)$$

where the coefficients α and β are given by Madsen. From this we can immediately deduce that the bulk viscosity becomes less important at both very low and very high temperatures. For low temperatures we find that

$$t_{bv}^{\text{low}} \approx 7.9 M_{1.4}^2 R_{10}^{-4} P_{-3}^2 T_9^{-2} m_{100}^{-4} \text{ s} \quad (104)$$

where m_{100} represents the mass of the strange quark in units of 100 MeV. Meanwhile, at high temperatures we cannot readily write down an expression with the appropriate dependence on M , R , P etcetera. Instead, we have to perform numerical calculations for each stellar model. Typical results (for canonical parameters) are shown in Figure 7. From this figure we immediately see that the r -mode instability would not - contrary to the case for neutron stars - be active in strange stars with a core temperature of 10^9 K. In a strange star the r -modes are unstable at lower temperatures (between $10^5 - 5 \times 10^8$ K) and also at temperatures above a few times 10^9 K. This is interesting since it means that the instability would be active for a brief period after a strange star is born. Then the mode would become stable until thousands of years later when the star has cooled sufficiently to enter the low-temperature instability window.

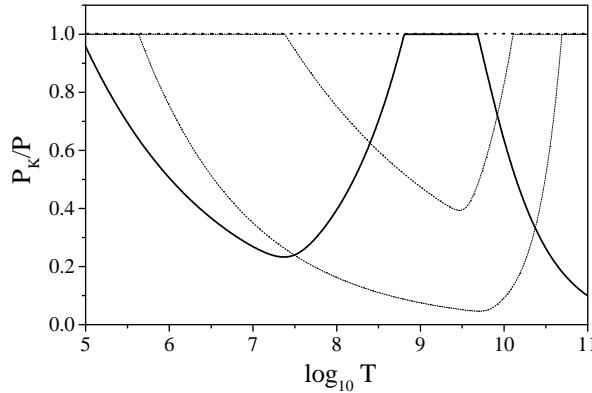


Fig. 7. The r -mode instability window for a strange star (solid line). As comparison we show the corresponding instability results for normal fluid and crusted neutron stars (dashed curves) from Figure 6. The decreased importance of bulk viscosity in hot strange stars is notable.

Similar estimates would be valid for a neutron star with a quark core, a so-called hybrid star⁹⁸ and Madsen⁹¹ has commented on the viscosity of some rather exotic states of matter that may be relevant for hybrid star cores.

5. Is the r -mode instability astrophysically relevant?

In the previous sections we have described the nature of the r -modes of a rotating neutron star. We have discussed the associated gravitational-wave driven instability and shown that it may operate in a neutron star with core temperature in a window between roughly $10^5 - 10^{10}$ K. This means that the instability should be relevant for hot young neutron stars that are born spinning at a rate above perhaps 5% of the dynamical Kepler limit.

We now want to discuss the possible astrophysical consequences of the r -

mode instability. By necessity this means that we will extrapolate the available results considerably. Astrophysical neutron stars are much more complex than the simple models for which the properties of the *r*-modes have so far been studied. Nevertheless, the speculations that we describe in this section are interesting and potentially of great importance. After all, they may eventually turn out to be largely correct! We must, of course, keep in mind that our understanding of much of the involved physics is (at best) rudimentary.

5.1. A phenomenological spin-evolution model

If we want to make the discussion quantitative rather than qualitative we must model the actual evolution of an *r*-mode as it grows and spins down the star. Given the current models, this is a challenging task and any model we devise is likely to be questionable. Still, it is important that we try to gain insight into this issue. After all, once we have an initial prediction we can try to identify the crucial elements of the model and work to improve on them. It is also relevant to address many important astrophysical questions at an “order of magnitude” level.

While the early growth phase of an unstable mode can be described by linear theory, an understanding of many effects (such as coupling between different modes) that will become important and may eventually dominate the dynamics require a nonlinear calculation. While such studies are outstanding we can only try to capture the essential features of the behaviour. Intuitively one would expect a growth of an unstable mode to be halted at some amplitude. As the mode saturates it seems plausible that the excess angular momentum will be radiated away and the star will spin down. This general picture is supported by detailed studies of instabilities in rotating ellipsoids^{92,93,94,95}.

We model the evolution of a spinning star governed by the *r*-mode instability using the phenomenological two-parameter model devised by Owen et al.¹⁵ That is, we consider the spin rate Ω and the mode-amplitude α as our key quantities. Recall that the latter is defined by expressing the velocity field (measured by an observer that is co-rotating with the star) as (for $l = m$ modes)

$$\delta\vec{u} \approx \alpha\Omega R \left(\frac{a}{R}\right)^l \vec{Y}_{ll}^B e^{i\omega_rt} \quad (105)$$

Assume that the total angular momentum of the system can be decomposed as

$$J = I\Omega + J_c \quad (106)$$

where the first term represents the bulk rotation of the star and the second is the canonical angular momentum of the *r*-mode. This representation makes sense since J_c corresponds to the second order change in angular momentum [‡]

[‡]To be more precise: There is an ambiguity in deciding whether J_c is the second-order change in angular momentum, because that second-order change also includes an arbitrary second

due to the presence of the mode^{49,50} (in absence of viscosity or radiation). In the case of r -modes we have

$$J_c = -\frac{3\Omega\alpha^2\tilde{J}MR^2}{2} \quad (107)$$

where we have defined

$$\tilde{J} = \frac{1}{MR^4} \int_0^R \rho a^6 da = \begin{cases} 3/28\pi & n=0 \\ 1.635 \times 10^{-2} & n=1 \end{cases} \quad (108)$$

We now assume that angular momentum is only dissipated via gravitational waves. The gravitational-wave luminosity follows from the $l=m=2$ current multipole formula, and we get

$$\dot{J}_{\text{gw}} = 3\Omega\alpha^2 \frac{\tilde{J}MR^2}{t_{\text{gw}}} \quad (109)$$

Taking a time-derivative of (106) and defining

$$\tilde{I} = I/MR^2 = \begin{cases} 2/5 & n=0 \\ 0.261 & n=1 \end{cases} \quad (110)$$

we have

$$\left(\tilde{I} - \frac{3}{2}\alpha^2\tilde{J} \right) \frac{d\Omega}{dt} - 3\Omega\alpha\tilde{J} \frac{d\alpha}{dt} = \frac{1}{MR^2} \dot{J}_{\text{gw}} \quad (111)$$

Here we have neglected the dependence of the moment of inertia on the spin rate (terms of order Ω^2). It is easy to include such terms and show that they have a minor effect.

A second equation follows from the expression for the energy of the mode (as measured in the rotating system), cf. (58),

$$E = \frac{1}{2}\alpha^2\Omega^2MR^2\tilde{J} \quad (112)$$

and the fact that the mode grows according to

$$\frac{dE}{dt} = -2E \left(\frac{1}{t_{\text{gw}}} + \frac{1}{t_{\text{diss}}} \right) \quad (113)$$

where

$$\frac{1}{t_{\text{diss}}} = \frac{1}{t_{\text{sv}}} + \frac{1}{t_{\text{bv}}} + \text{other dissipation terms} \quad (114)$$

order addition of an solution to the time-independent linearized equations that simply adds differential rotation to the equilibrium star. This ambiguity can be resolved by requiring that the perturbation be canonical and that the Lagrangian displacement have no second-order part^{49,50}. Under these conditions J_c is the second order change in the angular momentum. For a growing mode, requiring that the second-order part grows exponentially in the manner $e^{(t/\tau)}$, (with τ the growth time of the mode energy) eliminates the ambiguity in a physically more appropriate way, but the consistent solution requires one to include the also the second order radiation-reaction term.

These two equations combine to give

$$\Omega \frac{d\alpha}{dt} + \alpha \frac{d\Omega}{dt} = -\alpha\Omega \left(\frac{1}{t_{\text{gw}}} + \frac{1}{t_{\text{diss}}} \right) \quad (115)$$

We can now combine (111) and (115) to get “evolution equations” for the mode amplitude and the bulk rotation rate:

$$\frac{d\alpha}{dt} = -\frac{\alpha}{t_{\text{gw}}} - \alpha \frac{A_-}{A_+} \frac{1}{t_{\text{diss}}} \quad (116)$$

and

$$\frac{d\Omega}{dt} = -\frac{1}{A_+} 3\alpha^2 \Omega \tilde{J} \frac{1}{t_{\text{diss}}} \quad (117)$$

where

$$A_{\pm} = \tilde{I} \pm \frac{3}{2} \alpha^2 \tilde{J} \quad (118)$$

These equations govern the mode-evolution in the phase when the amplitude of the *r*-mode grows, as well as in the late phase when the temperature has decreased sufficiently to again make the mode stable.

Since they should be adequately described by linear perturbation theory, the very early and late parts of the evolution of an unstable mode are comparatively well understood. Unfortunately, it is during the nonlinear phase that the main spin-down is anticipated to occur. Hence, we need to model the nonlinear regime in a “meaningful” way. As our starting point we adopt the simple view that the mode-amplitude will saturate at a critical value $\alpha_s \leq 1$. The assumed upper limit of the saturation amplitude, $\alpha_s = 1$ corresponds to the *r*-mode carrying a considerable part of the angular momentum of the system. We can readily estimate that

$$\left| \frac{J_c}{I\Omega} \right| \approx \frac{3\alpha^2 \tilde{J}}{2\tilde{I}} \approx 0.1\alpha^2 \quad (119)$$

Hence, a value of $\alpha_s \approx 1$ should be considered as very large (cf. also (51)). We assume that α stays constant throughout the saturated phase and that Ω evolves according to (111) with $\dot{\alpha} = 0$. This means that we have

$$\frac{d\Omega}{dt} = \frac{1}{A_-} \frac{3\Omega\alpha^2 \tilde{J}}{t_{\text{gw}}} \quad (120)$$

Since the viscosities are sensitive to temperature changes we need to model also the thermal evolution of the star in some suitably simple way. To do this we account for two main processes: i) cooling due to (say) the modified URCA process, and ii) reheating due to energy deposited from the mode to the heatbath via the shear viscosity. In stars with a solid crust we also include energy dissipated into heat in the Ekman layer. The bulk viscosity leads to the creation of neutrinos that can safely be assumed to escape from the star on a very short time-scale at the temperatures we consider.

We will typically initiate the evolution of the star at some temperature (recall that neutron stars are born above 10^{11} K and cool to 10^9 K in a few months). To relate the initial temperature to the thermal energy available in the heat bath we can use⁹⁶

$$E_T = 3.5 \times 10^{47} M_{1.4} \rho_{15}^{-2/3} T_9^2 \text{ erg} \quad (121)$$

Then we model the subsequent thermal evolution of the star by assuming that E_T evolves according to

$$\frac{dE_T}{dt} = -\dot{E}_\nu + \dot{E}_{\text{visc}} \quad (122)$$

where the luminosity due to the modified URCA process is⁹⁶

$$\dot{E}_\nu^{\text{URCA}} = 1.1 \times 10^{40} M_{1.4} \rho_{15}^{1/3} T_9^8 \text{ erg/s} \quad (123)$$

This mechanism is anticipated to dominate the cooling in the relevant range of temperatures, but one should not forget the possibility of more rapid cooling due to for example the direct URCA reactions⁶⁴. The possible outcomes should a faster cooling mechanism be operating have been discussed by Andersson, Kokkotas and Schutz³⁸ and Yoshida et al.⁹⁷

The reheating due to shear viscosity is readily determined from

$$\dot{E}_{\text{sv}} = \frac{2\alpha\Omega^2 M R^2 \tilde{J}}{t_{\text{sv}}} \quad (124)$$

The heat generated due to the presence of an Ekman layer in an older neutron star with a crust follows from a similar expression.

5.2. The spin-evolution of young neutron stars

Given the general features of the r -mode instability we expect that it could be relevant in newly born neutron stars. Hence it is interesting to consider the observational data regarding the initial spin rate of young pulsars. The best studied young pulsar is the Crab (PSR0531+21), whose initial period is estimated (assuming the standard magnetic braking model) to have been about 19 ms⁹⁸. Even the recently discovered young 16 ms X-ray pulsar in the supernova remnant N157B probably had an initial period no shorter than a few ms⁹⁹ (assuming a braking index typical of young pulsars). These estimates are in clear contrast with the shortest known period of a recycled pulsar of 1.56 ms, and with the theoretical lower limit on the period of about 0.5 to 2 ms¹⁴, depending on the equation of state. The available data essentially suggests that neutron stars are formed spinning slowly, at perhaps $\Omega/\Omega_K < 0.1$. From a theoretical point of view, this is rather surprising since, assuming that angular momentum is conserved in the collapse event that forms the neutron star, young pulsars ought to be spinning close to the Kepler limit.

One possible explanation for the slow initial spin rate of newly born neutron stars was proposed by Spruit and Phinney¹⁰⁰, who argue that magnetic locking

between the core and the envelope of the progenitor star may prevent the collapsing core from spinning rapidly. Their estimates suggest that the core would actually rotate far too slowly to lead to (by conservation of angular momentum) young neutron stars spinning as fast as they do. To explain the observed rotation rates, Spruit and Phinney propose that the neutron star spin is due to the birth kicks that also (or alternatively) produce the large linear velocities observed for pulsars.

Even if correct, Spruit and Phinney's model still suggests that some pulsars are born spinning rapidly. Theoretically, it is easy to estimate that birth kicks that lead to velocities larger than 500 km/s can also produce rotation at (or above) the Kepler limit. Furthermore, the 16 ms pulsar in N157B is clear evidence that some pulsars form with periods shorter than say 10 ms. For such young neutron stars, the secular *r*-mode instability may play a role in determining the rotation rate. It is this possibility that we focus on here.

As soon as we bring the spin-evolution model of the previous section to bear on the problem we see that the key parameter is the saturation amplitude α_s . Provided that α_s is sufficiently large, the *r*-modes will spin a young neutron star down appreciably. If we assume that the neutron star is born with core temperature well above 10^{10} K and that it initially spins at the Kepler limit, the *r*-mode instability comes into play within a few seconds as the star cools and enters the instability window. The mode then grows from some small initial amplitude to the saturation level in a few minutes. Once the mode has saturated, the star spins down. At some point the star has cooled (or spun down) sufficiently that the *r*-mode is again stable. Then the mode amplitude decays and the star presumably enters a phase where magnetic braking takes over and dominates the spin-evolution. Examples of this scenario are shown in Figure 8.

The final spin period depends on several factors. The most important of these are the saturation amplitude, the cooling rate and whether a crust forms during the evolution. If we consider a simple perfect fluid model the *r*-mode spin-down leads to $P \approx 12 - 22$ ms for α_s in the range 0.01 – 1 (and canonical neutron star parameters). This result is clearly consistent with observations for many young pulsars (in particular the Crab).

As we discussed in Section 4.5, the *r*-mode instability may be strongly suppressed in neutron stars with a solid crust. Given that the melting temperature of the crust could be as high as 10^{10} K, the crust may form shortly after the neutron star is born and we need to discuss the effect that this may have on the *r*-mode scenario. The interplay between a growing mode and the formation of the crust leads to complicated questions that require much further study. For example, it is not at all clear to what extent the melting temperature of the crust is affected by large scale surface waves in the star. Here we will not address such questions in detail (but see comments in Section 6). Instead we consider what may be a “worst case scenario” as far as the instability is concerned. We

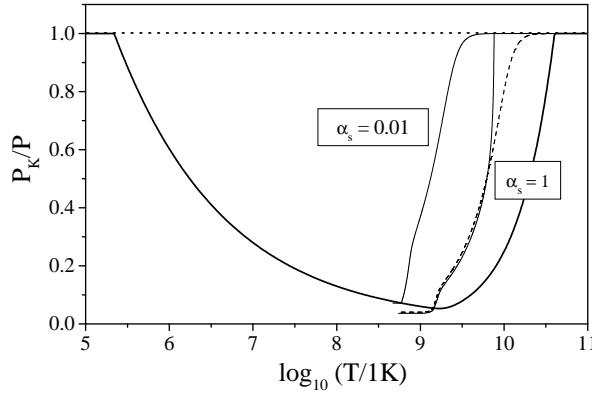


Fig. 8. Period evolutions resulting from the *r*-mode spin-down scenario. We assume that the star is isothermal and that it cools entirely due to the modified URCA process. Two different evolutions corresponding to an *r*-mode saturated at $\alpha_s = 0.01$ and 1, respectively, are shown as solid lines. The dashed curve indicates the evolution that would follow if the *r*-mode was initially excited to the saturation level rather than given a very small initial amplitude ($\alpha = 10^{-6}$) at the onset of the instability. It is notable the initial amplitude has little effect on the period at the end of the spin-down phase.

assume that the *r*-mode will not be able to prevent the crust from forming (and that the melting temperature is at the high end of the anticipated range). The main dissipation mechanism is then due to the presence of the Ekman layer (at least above the superfluid transition temperature). Resulting spin evolutions are shown in Figure 9. In this scenario, the period reached after the spin-down phase lies in the range $P \approx 2.5 - 4.5$ ms. Just as in the perfect fluid case, this is an interesting prediction since it agrees quite well with the data for the recently discovered 16 ms pulsar PRS J0537-6910 in N157B.

We have thus seen that the *r*-mode instability can lead to young neutron stars being spun down to rotation rates that would agree quite well with extrapolations from current observations. Furthermore, we have predicted that hot young neutron stars may follow two different evolution routes¹⁰¹. Which scenario applies depends sensitively on the early cooling of the star and the crustal formation temperature. To illustrate this, we consider the evolution of a neutron star just after its birth in a supernova explosion. We might expect to model its *r*-mode evolution using the normal (crust-free) fluid viscous damping times for stellar temperatures above the melting temperature of the crust (T_m), and the viscous boundary layer damping time for temperatures below T_m . However, the situation is probably a little more complicated than this. Recall that the latent heat (i.e. the Coulomb binding energy) of a typical crust is

$$E_{\text{lat}} \approx 10^{48} \text{ erg} \quad (125)$$

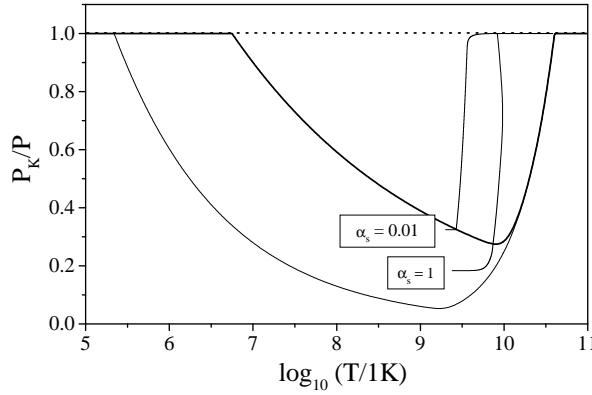


Fig. 9. Spin-down evolutions for a neutron star in which a solid crust forms before the *r*-mode has grown appreciably. The presence of an Ekman layer at the base of the crust leads to a strong dissipation of the mode energy and the final spin-period after the instability phase is much shorter than in the crustless case.

As is clear from (58) the energy in the *r*-mode (which grows exponentially on a timescale $t_{\text{gw}} \sim 20 - 50$ s) will easily exceed E_{lat} , provided that the time taken for the star to cool to T_m is sufficiently long. This would then prevent the formation of the crust, even when $T < T_m$. Then the star will spin down in the manner shown in Figure 8. This phase will end either because the star leaves the instability window or because the mode energy has fallen below the crustal binding energy. Supposing that the *r*-mode energy is distributed in such a way that (say) 10% is located in the region where the crust will form (the outer km or so of the star), we reach balance, i.e. $E_{\text{lat}} \approx 0.1E_{\text{mode}}$, when

$$P_{-3} \approx 10\alpha_s \quad (126)$$

In other words, for a saturation amplitude of order unity the crust would be unable to form until the star has spun down to a rotation rate of roughly 100 Hz. Again, this would be consistent with the extrapolated initial spin rates of many young pulsars. On the other hand, if the mode is not given time to grow very large it will not prevent crust formation at T_m . The *r*-mode instability would spin the star down as in Figure 9. A likely key parameter (in addition to the cooling rate, the melting temperature and the extent to which a large amplitude mode can melt the crust, see section 6) is whether the supernova collapse leads to a large initial *r*-mode amplitude or not.

Besides explaining the slow rotation of young pulsars, the *r*-mode instability has the consequence that young neutron stars can only reach rotation periods shorter than (say) 3-5 ms if they are recycled by accretion in a binary system. The alternative model, that these stars are formed by accretion-induced collapse

of a white dwarf, is inconsistent with the r -mode scenario³⁸. The collapse would form a star hot enough to spin down because of the instability. However, the situation would be quite different in the case of strange stars⁹⁰. As shown in Figure 7, the instability window for strange matter differs significantly from the neutron star one. This means that the r -mode instability would be less effective (apart from at very high temperatures) for strange stars. Furthermore, a strange star cools much slower than a neutron star. Assuming that the strange star cools on a timescale $t \approx 10^{-4}T_9^{-4}$ yrs, we find that the strange star reaches $P \approx 3$ ms after 4×10^5 yrs. This means that an observation of a young millisecond pulsar spinning faster than (say) 3 ms may indicate the presence of a strange star⁹⁰.

5.3. Are the gravitational waves observable?

Having suggested that the r -mode instability may spin a newly born neutron star down to a fraction of its initial spin rate in a few months we want to know whether the gravitational waves that carry away most of the stars initial angular momentum are detectable. This is a particularly relevant question given the generation of large-scale interferometers that is about to come into operation¹.

We assess the detectability of the emerging gravitational waves in the standard way. First of all, we note that the frequency of the emerging gravitational waves is (for the main $l = m = 2$ r -mode)

$$f_{\text{gw}} = \frac{2\Omega}{3\pi} . \quad (127)$$

From the gravitational-wave luminosity we readily deduce that the dimensionless strain amplitude follows from¹⁵

$$h(t) = 7.54 \times 10^{-23} \alpha \tilde{J} M_{1.4} R_{10}^3 \left(\frac{15 \text{ Mpc}}{D} \right) \quad (128)$$

where D is the distance to the source, here assumed to be in the Virgo cluster. At this distance one would expect to see several neutron stars being born per year. The typical r -mode scenarios shown in Figures 8 and 9 then lead to the gravitational-wave strains illustrated in Figure 10. Clearly, this amplitude is not sufficiently strong that it can be observed without a detailed data analysis strategy. To assess the possible improvement that would follow if such a strategy could be developed for the r -mode signal we will use the standard matched-filtering approach. This gives us an idea of the improvement that a specially tailored data analysis approach may bring to the r -mode detection problem, even though it must be recognized that matched filtering is unlikely to be possible for this kind of signals^{102,103}.

With the Fourier transform of the gravitational-wave signal defined by

$$\tilde{h}(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi f t} dt \quad (129)$$

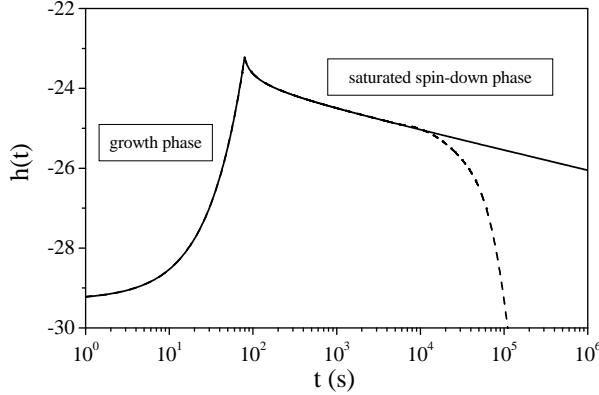


Fig. 10. The dimensionless gravitational-wave amplitude following from a typical *r*-mode scenario. The two phases of evolution (the early growth phase of the mode and the spin-down after saturation) are easily distinguished. The second phase is the most likely to be observed with future detectors. We show signals corresponding to both fluid neutron stars (thick solid line) and ones with a crust (dashed). In both cases we assume that the saturation amplitude is $\alpha_s = 1$. It should be noted that the signal lasts 10^7 s in the fluid case but only 10^5 s for the crusted star.

the signal-to-noise ratio attainable by optimal filtering can be determined from

$$\left(\frac{S}{N}\right)^2 = 2 \int_0^\infty f \frac{|\tilde{h}|^2}{h_{rms}^2} df \quad (130)$$

where $h_{rms} = \sqrt{f S_h(f)}$, S_h being the spectral density of the strain noise in the detector. Now introducing the characteristic amplitude h_c we can write

$$\left(\frac{S}{N}\right)^2 = 2 \int_0^\infty \frac{df}{f} \left(\frac{h_c}{h_{rms}}\right)^2 \quad (131)$$

where

$$h_c = f |\tilde{h}| \approx h \sqrt{f^2 \left| \frac{dt}{df} \right|} \quad (132)$$

The last relation follows via the stationary phase approximation. It can be interpreted as meaning that the detectability of an almost periodic signal improves as the square root of the number of cycles radiated in the time it takes the frequency to change by f . In Figure 11 we compare this characteristic amplitude to estimated sensitivity curves for the new generation of interferometric detectors. We immediately see that the *r*-mode instability would be a promising source for an advanced generation of detectors such as LIGO II. One can estimate that the gravitational waves from a hot young neutron star will be detectable with a good signal-to-noise ratio^{15,102,103}. This makes the *r*-mode instability one of the

most promising sources of detectable gravitational waves. It is unlikely that the r -modes will be observed by the first generation detectors, however. For this to happen requires a unique event in our Galaxy or the local group. One would typically expect a supernova in the galaxy every 30 years or so, which means that we would be extremely lucky to see such an event. Anyway, the chances of future detection are promising and they may improve as our understanding of the instability becomes more detailed and better theoretical templates for the gravitational-wave signal can be constructed.

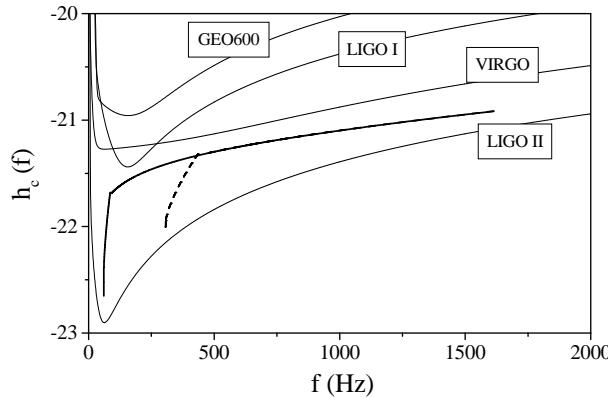


Fig. 11. The effective amplitude achievable after matched filtering (which is somewhat unrealistic for the r -mode instability) compared to the expected sensitivity of the new generation of interferometric detectors. The two gravitational-wave signals are the same as in Figure 10.

In addition to signals from individual neutron stars, there will be a stochastic background of gravitational waves from cosmological neutron stars spinning down due to the instability. Estimates show that this background will be difficult to detect, but it could perhaps be relevant for an advanced generation of detectors^{15,104}.

6. Accreting neutron stars

In the last few years the Rossi X-ray Timing Explorer has provided a wealth of observational data regarding accreting neutron stars. These observations present a challenge for theorists in that neutron stars in low-mass X-ray binaries (LMXB) seem to be confined to a rather narrow range of spin frequencies, perhaps 260-590 Hz¹⁰⁵. Different models have been proposed to explain this surprising result. The first model (due to White and Zhang¹⁰⁶) is based on the standard magnetosphere model for accretion induced spin-up, while two other models are inspired by the idea that gravitational radiation balances the accretion torque. In the first such model for the LMXB (proposed by Bildsten¹⁰⁷ and

recently refined by Ushomirsky, Cutler and Bildsten¹⁰⁸), the gravitational waves are due to an accretion induced quadrupole deformation in the deep neutron star crust. The second gravitational-wave model relies on the *r*-mode instability to dissipate the accreted angular momentum from the neutron star¹⁰⁹.

The idea that gravitational waves from unstable *r*-modes provide the agent that balances the accretion torque was first analysed in detail by Andersson, Kokkotas and Stergioulas¹⁰⁹ (but see also Bildsten¹⁰⁷). Originally, it was thought that an accreting star in which the *r*-modes were excited to a significant level would reach a spin-equilibrium, very much in the vein of early suggestions by Papaloizou and Pringle¹¹⁰ and Wagoner¹¹¹. Should this happen, the neutron stars in LMXB would be prime sources for detectable gravitational waves. That the associated (essentially periodic) gravitational waves would be detectable can be seen in the following way: Assume that the *r*-mode instability is active and provides a limit on the spin of Sco X1, which is the strongest X-ray source in the sky and therefore a prime candidate for this kind of speculation. If we assume that the average accretion rate onto the neutron star is $\dot{M} \approx 3 \times 10^{-9} M_{\odot}/\text{yr}$ and that the accretion torque is balanced by gravitational radiation we deduce that $h \approx 3.5 \times 10^{-26} \rightarrow h_c \sim 10^{-21}$ after two weeks worth of signal has been accumulated¹⁰⁹.

However, this idea is probably not viable[§] In addition to generating gravitational waves that dissipate angular momentum from the system, the *r*-modes will heat the star up (via the shear viscosity and the Ekman layer). Since the viscosity gets weaker as the temperature increases, the mode-heating triggers a thermal runaway^{113,114} and in a few months the *r*-mode would spin an accreting neutron star down to a rather low rotation rate. This essentially rules out the *r*-modes in galactic LMXB as a source of detectable gravitational waves, since they will only radiate for a tiny fraction of the systems lifetime.

This mechanism could still be of great astrophysical significance¹⁰¹. Hence, we want to model how the potential presence of an unstable *r*-mode affects the spin-evolution of rapidly spinning, accreting neutron stars. To do this we use the phenomenological two-parameter model described in section 5.1. Then the following picture emerges: After accreting and spinning up for something like 10^7 years, the star reaches the period at which the *r*-mode instability sets in. For a canonical neutron star this corresponds to a period of 1.5 ms (at a core temperature of 10^8 K). It is notable (albeit likely coincidental) that this value is close to the 1.56 ms period of the fastest known pulsar PSR1937+21. Once the *r*-mode becomes unstable (point A in Figure 12), viscous heating (mainly due to the energy released in the Ekman layer) rapidly heats the star up to a few times 10^9 K. The *r*-mode amplitude increases until it reaches the saturation level (amplitude α_s), at which unspecified nonlinear effects halt further growth

[§]Although it should be noted that the Wagoner scenario might be relevant if accretion at extreme rates is possible. The role of the unstable *r*-modes during such hypercritical accretion has been discussed by Yoshida and Eriguchi¹¹².

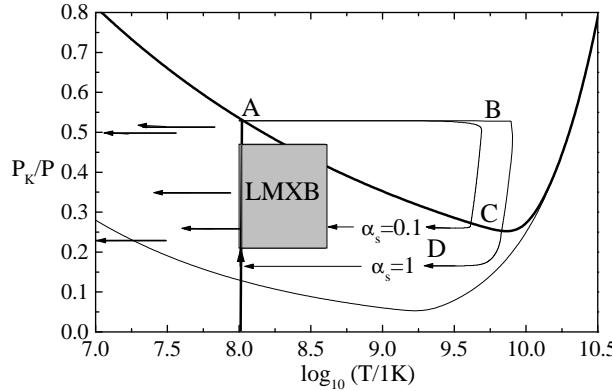


Fig. 12. The r -mode instability window relevant for old neutron stars. We show results for the simplest (crust-free) model (thin solid line), as well as for a star with a crust (thick solid line). We illustrate two typical r -mode cycles (for mode saturation amplitudes $\alpha_s = 0.1$ and 1), resulting from thermo-gravitational runaway after the onset of instability. Once accretion has spun the star up to the critical period (along the indicated spin-up line (thick vertical line)) the r -mode becomes unstable and the star evolves along the path A-B-C-D. After a month or so, the mode is stable and the star will cool down until it again reaches the spin-up line. For comparison with observational data, we indicate the possible range of spin-periods inferred from current LMXB data (shaded box) as well as the observed periods and estimated upper limits of the temperature of some of the most rapidly spinning millisecond pulsars (short arrows).

(point B in Figure 12). Once the mode has saturated, the neutron star rapidly spins down. When the star has spun down to the point where the mode again becomes stable (point C in Figure 12), the amplitude starts to decay and the mode eventually plays no further role in the spin evolution of the star (point D in Figure 12). Two examples of such r -mode cycles (corresponding to $\alpha_s = 0.1$ and 1, respectively) are shown in Figure 12.

This simple model leads to some interesting quantitative predictions. First of all, the model suggests that an accreting star will not spin up beyond 1.5 ms. This value obviously depends on the chosen stellar model, but it is independent of the r -mode saturation amplitude and only weakly dependent on the accretion rate (through a slight change in core temperature). In fact, the accretion rate only affects the time it takes the star to complete one full r -mode cycle. As soon as the mode becomes unstable the spin-evolution is dominated by gravitational radiation and viscous heating. Once the star has gone through the brief phase when the r -mode is active it has spun down to a period in the range 3-5 ms (corresponding to $0.01 \leq \alpha_s \leq 1$). At this point the mode is again stable and continued accretion may resume to spin the star up. Since the star must accrete roughly $0.1M_{\odot}$ to reach the instability point, and the LMXB companions typically have masses in the range $0.1 - 0.4M_{\odot}$, it can pass through several “ r -

mode cycles” during its lifetime.

We can readily confront these results with current observations[¶]. We note that the model predicts that an accreting neutron star must, once it has been spun up beyond (say) 5 ms, remain in the rather narrow range of periods 1.5 – 5 ms until it has stopped accreting and magnetic braking slows it down. Since a given star can go through several *r*-mode cycles before accretion is halted one would expect most neutron stars in LMXB and the millisecond pulsars to be found within this range. As is clear from Figure 12, this prediction agrees well with the range of rotation periods inferred from observed kHz quasiperiodic oscillations in LMXB. The observed range shown in Figure 12 corresponds to rotation frequencies in the range 260–590 Hz¹⁰⁵. The model also agrees with the observed data for millisecond pulsars¹⁰¹, which are mainly found in the range 1.56 – 6 ms. In other words, the *r*-mode runaway model is in agreement with current observed data for rapidly rotating neutron stars.

However, many crucial pieces of physics have not yet been included in this model. For example, we based the discussion on the simplest estimate (97) of the Ekman layer dissipation, rather than more refined ones attempting to include the crust–core interaction in a detailed way^{84,85}. Furthermore, we assumed that the heat released in the Ekman layer will not be able to melt the crust. This latter issue has recently been studied in some detail by Lindblom, Owen and Ushomirsky¹¹⁶ (see also Wu, Matzner and Arras¹¹⁷), who solve the appropriate heat equation incorporating thermal conductivity for the core fluid and the crust as well as neutrino cooling due to the modified URCA process. They find that, for a star spinning at the Kepler limit, an *r*-mode with an amplitude exceeding $\alpha \approx 5 \times 10^{-3}$ will be able to melt the crust. This means that if the *r*-mode instability is active in a young neutron star, the formation of the crust may be significantly delayed (as we argued in section 5.2). But it is not yet clear what will happen if the *r*-modes do indeed manage to melt an existing crust. As soon as the crust melts the Ekman layer (that led to the excessive heating) disappears and the material will rapidly cool down to a level where the crust would begin to form again. As argued by Lindblom et al¹¹⁶, the likely outcome is a solid-fluid mixed state which will be very difficult to model in detail.

We conclude this section by discussing briefly the detectability of the gravitational waves that are radiated during the relatively short time when the *r*-mode is saturated and the star spins down. Since the *r*-mode is active only for a small fraction of the lifetime of the system (something like 1 month out of the 10^7 years it takes to complete one full cycle) the event rate is far too low to make galactic sources relevant. However, it is interesting to note that the spin-evolution is rather similar to that of a hot young neutron star once the *r*-mode has reached its saturation amplitude. This means that we can analyse the detectability of the emerging gravitational waves using the framework of section 5.3. We then

[¶]Here it is worth mentioning that Brown and Ushomirsky¹¹⁵ have discussed how observations of X-ray transients may provide a constraint on the *r*-mode amplitude in such systems.

find that these events can be observed from rather distant galaxies. For a source in the Virgo cluster these gravitational waves could be detected with a signal to noise ratio of a few using LIGO II. However, even at the distance of the Virgo cluster these events would be quite rare. By combining a birth rate for LMXB of 7×10^{-6} per year per galaxy with the fact that the the volume of space out to the Virgo cluster contains $\sim 10^3$ galaxies, and the possibility that each LMXB passes through (say) four r -mode cycles during its lifetime we deduce that one can only hope to see a few events per century in Virgo. In order to see several events per year the detector must be sensitive enough to detect these gravitational waves from (say) 150 Mpc. This would require a more advanced detector configuration.

7. Steps towards more detailed modelling

From the discussion in the previous sections it should be clear that the present results indicate that the unstable r -modes may have great astrophysical significance, and that the associated gravitational-wave signals may well be detected within a few years. These ideas are obviously very exciting, but it is important to realize that our modelling must be improved in many directions before any reliable conclusions can be drawn. In this section we discuss some recent work that provide extensions (or alternatives) to the ideas that we have already described.

7.1. Inertial modes and rapid rotation

As was mentioned in the introduction, the r -modes belong to a larger class of pulsation modes which in fluid mechanics are commonly described as “inertial modes”⁸². These modes are all dominated by the Coriolis force and for slowly rotating stars their frequencies scale linearly with Ω . In non-isentropic stars the rotation dominated modes are all toroidal in nature (the r -modes). In contrast, the space of zero frequency modes of isentropic spherical stars includes the spheroidal g -modes as well as the r -modes. This large degenerate subspace of zero-frequency modes is then split by rotation, and the inertial modes of isentropic stars generally have a “hybrid” nature³⁶ (their spherical limit is a mixture of toroidal and spheroidal perturbations).

Detailed calculations of such hybrid modes in Newtonian stars have been made by Lockitch and Friedman³⁶ as well as Yoshida and Lee^{118,119} (see also Lee et al¹²⁰ and the studies of Maclaurin spheroids discussed below). These studies show that the low frequency mode spectrum of a rotating star is tremendously rich. Furthermore, many of the hybrid modes are affected by the CFS instability mechanism. Estimated growth times and viscous damping rates for these unstable modes suggest that they are likely to lead to weaker instabilities than that of the fundamental r -mode [¶]. This can be understood from the fact

[¶]In this context, we should also mention that instabilities associated with the g -modes in

that a mode with a significant spheroidal part will be affected by (for example) bulk viscosity to a larger extent.

A particularly interesting question, that provided the original motivation for the work of Lockitch and Friedman³⁶, concerns what happens to all the *r*-modes of a non-isentropic star as $\mathcal{A}_s \rightarrow 0$, i.e. in the isentropic limit. Recall that a non-isentropic star has an infinite set of *r*-modes for each l and $m \neq 0$, while an isentropic star retains only a single such mode for $l = m$. The general consensus at the present time is that most of the non-isentropic *r*-modes will change their nature by acquiring a spheroidal component as $\mathcal{A}_s \rightarrow 0$, i.e. they become hybrid modes. Some light on the detailed behaviour has recently been shed by Yoshida and Lee¹¹⁹. They suggest that the *r*-modes of a non-isentropic star fall into one of three categories. The first class are the fundamental $l = m$ modes (that have eigenfunctions $T_{ll} \sim a^{l-1}$ and lead to the strongest instability). These modes are largely unaffected by changes in \mathcal{A}_s . Similar results were reported by Andersson, Kokkotas and Schutz³⁸. The second class of modes are the various $l = m$ overtones (whose eigenfunctions have nodes inside the star). According to Yoshida and Lee¹¹⁹, these modes approach hybrid modes as the spin-rate of the star increases to a level where the slow-rotation approximation is no longer appropriate. Finally, the $l \neq m$ *r*-modes of a non-isentropic star have frequencies that vanish (in the rotating frame) as the spin-rate increases. Consequently, these modes may not be relevant in a rapidly rotating star.

It turns out that the general behaviour of the modes in the isentropic limit is rather similar to what happens at fast rotation rates. Until recently, low-frequency modes had not been calculated for rapidly rotating stars. The situation has changed with calculations for the Maclaurin spheroids¹²² (essentially reproducing and elucidating very early work by Bryan¹²³) as well as for rapidly rotating polytropes⁹⁷. These calculations are complicated by the fact that the mode structure changes as the spin rate increases, and for rapidly rotating stars one must include a large number of multipoles in a description of the mode eigenfunctions. Interestingly, the polytrope results of Yoshida et al.⁹⁷ indicate that the slow rotation expansion (taken to order Ω^2) provides a good representation of the *r*-modes also in rapidly spinning stars. This means that the current models for the spin evolution of young neutron stars etcetera may, in fact, be better than expected.

7.2. Slowly rotating stars in general relativity

One important step towards improving our understanding of the *r*-mode instability corresponds to describing the modes in general relativity. The Newtonian picture of stellar pulsation, that we introduced in sections 2.1-2.3, is readily generalised to the relativistic case. The main difference is that the gravitational waves generated by the modes now appear as perturbations of the spacetime

non-isentropic stars have been discussed by Lai¹²¹.

metric. Consequently, the various modes are no longer “normal modes”, but must satisfy outgoing-wave boundary conditions at spatial infinity. This provides an additional complication in general mode-calculations (where the eigenfrequencies are now complex), but it is not relevant for a slow-rotation study of low-frequency oscillations. As can be deduced from (65) the gravitational-wave dissipation will not enter until at very high orders in the slow-rotation expansion. Hence, the r -mode frequencies remain real to leading order, and we are (in principle) dealing with a “standard” normal-mode problem.

Considerable progress towards an understanding of the relativistic r -mode problem has been made recently^{5,124,125,126,127}. Particularly relevant is work by Lockitch, Andersson and Friedman¹²⁸ that puts the various facets of the problem in the appropriate context and provides the first actual calculation of the relativistic r -modes. The picture that emerges with this work is somewhat more complex than the Newtonian one, but it can be understood if we recall the difference between isentropic and non-isentropic Newtonian perturbations. First of all, one can prove that (apart from a set of stationary dipole modes¹²⁹) rotating relativistic isentropic stars have no pure r -modes (modes whose limit for a spherical star is purely toroidal). This is in contrast with the isentropic Newtonian stars which retain a vestigial set of purely toroidal modes (the fundamental $l = m$ modes that lead to the strongest instability). Instead, the relativistic corrections to the Newtonian r -modes with $l = m \geq 2$ are both toroidal and spheroidal. Thus these modes become discrete hybrid modes of the corresponding relativistic models. So far these modes have been computed for slowly rotating isentropic stars to first post-Newtonian order ($\sim M/R$).

As in the Newtonian problem, non-isentropic stars are somewhat different. In the slow-motion approximation in which they have so far been studied, non-isentropic stars have, remarkably, a continuous spectrum. Kojima¹²⁴ has shown that purely axial modes would be described by a single, second-order differential equation. The continuous spectrum is implied by the fact that the corresponding eigenvalue problem is singular (the coefficient of the highest derivative term of the equation vanishes at some value of the radial coordinate). The mathematical arguments for the continuous spectrum have been elucidated by Beyer and Kokkotas¹²⁵. However, as the latter authors point out, it is not yet clear that the continuous spectrum is physically relevant. In addition to the possible continuous spectrum one can show that discrete r -mode solutions also exist. Such discrete mode-solutions to Kojima’s equation were recently calculated¹²⁸. These modes are the relativistic analogue to the Newtonian r -modes in non-isentropic stars.

Even though much work remains before the r -modes and their instability are understood in general relativity, the recent results provide much useful information. In particular, even though the nature of the equations for isentropic relativistic stars is rather different from the corresponding non-isentropic ones, the particular modes that would be the analogues of the vestigial $l = m$ r -modes

of isentropic Newtonian stars are not too dissimilar¹²⁸. In fact, the calculated mode frequencies differ only at the level of a few percent. The toroidal components of the velocity field are also virtually identical (although one must remember that the relativistic isentropic mode also has a significant spheroidal component).

7.3. *The role of the magnetic field*

Another important facet of neutron star physics that we have not yet discussed is the magnetic field. Intuitively one would expect the interplay between a large amplitude pulsation mode and the magnetic field to be interesting. The simple fact that this may lead to observable effects is strong motivation for trying to incorporate the magnetic field in our models. To do this is, however, far from trivial and at the present time all results in this direction must be considered as rather uncertain. On the other hand, the same is true for the *r*-mode spin-down scenario (section 5.1) so why should we shy away from speculating on the role of the magnetic field?

There have so far been three main discussions of magnetic fields in the context of the *r*-mode instability. In the first of these, Spruit¹¹³ outlined a possible scenario where an accreting neutron star with a weak magnetic field is spun up to the point where the *r*-mode becomes unstable (see Figure 12). As we discussed earlier, it is likely that the *r*-mode instability then leads to a thermo-gravitational runaway that heats the star up and makes the mode-amplitude grow. Spruit suggests that gravitational radiation reaction then induces differential rotation in the star. If so, the interior magnetic field could wind up until it reaches a critical point and becomes unstable due to buoyancy forces. When this happens, the star sheds a considerable amount of electromagnetic energy, perhaps as a gamma-ray burst, which slows it down significantly. Left over after the turmoil is a slowly rotating neutron star with an extreme magnetic field (of the order of $10^{14} - 10^{15}$ G): a magnetar^{8,9}. This idea is undoubtedly speculative but it raises some interesting questions, and may also capture some features of the *r*-mode runaway. Qualitatively, the most important issue regards whether the instability leads to differential rotation or not.

All the models that we have discussed in the previous sections assume that the star remains spinning uniformly. That this will actually be the case is far from clear. An important counterexample is the evolution of an unstable bar-mode in a rotating ellipsoid. As was first shown by Miller⁹³, an unstable mode drives a Maclaurin spheroid towards a Dedekind ellipsoid (more or less a rotating american football). En route the evolution is thought to proceed via a sequence of differentially rotating Riemann ellipsoids. By analogy one might therefore expect that the *r*-mode instability will generate differential rotation. To conclusively prove this will, however, not be easy since it involves modelling how radiation reaction affects various parts of the star.

The question whether an unstable r -mode can be prevented from growing by the magnetic field was recently discussed by Rezzolla, Lamb and Shapiro¹³⁰. In an interesting approach to the problem, they considered the leading order result for the r -mode velocity field (basically the time-derivative of our equations (48) and (49)) as evolution equations for the fluid elements. By evolving this system in time one finds a differential drift, the magnitude of which depends on the latitude of the fluid element and the mode amplitude. Rezzolla et al study how this deduced differential drift affects the magnetic field of the star (assuming that the field is frozen into the fluid elements). This leads to several suggestions. First of all, it is estimated that the r -mode oscillations cannot prevail if the star has a magnetic field $\sim 10^{16}$ G or higher. In a star with a weaker magnetic field, $B \geq 10^{10}$ G, the rotation rate must be above $0.35\Omega_K$ in order for the mode to survive. This means that the instability could operate both in young neutron stars ($B \sim 10^{12}$ G) and recycled ones ($B \sim 10^8$ G) provided that they spin fast enough. Secondly, the differential drift due to the r -mode twists the magnetic field and, just like in Spruit's model, this affects the strength and nature of the field. Rezzolla et al predict that, given an initial configuration with a strong poloidal field and a much weaker azimuthal one, the magnetic field evolves in such a way that after the r -mode spin-down the azimuthal field has been strengthened to the point where it is 4-8 orders of magnitude stronger than the (virtually unchanged) poloidal field. This would suggest that the r -mode instability generates strong azimuthal magnetic fields in young pulsars. This is an intriguing possibility, and it may be worthwhile investigating whether it could be tested observationally.

These are interesting suggestions, but they come with a disclaimer. The derived differential drift is not determined consistently within perturbation theory. Rezzolla et al. use first order (in both α and Ω) results to deduce a nonlinear effect (of order α^2). In absence of true nonlinear calculations it is not clear to what extent the predicted drift will be present.

Several other issues regarding the possible role of the magnetic field are discussed in a recent study by Ho and Lai¹³¹. First of all, they discuss the effect of the standard magnetic braking (that dominates the spin-down in absence of the unstable r -modes). From the standard results one finds that magnetic braking proceeds on a timescale

$$t_B \approx \frac{6c^3 I}{B^2 R^6 \Omega^2} = 3 \times 10^8 M_{1.4} R_{10}^{-4} B_{12}^{-2} P_{-3}^2 \text{ s}, \quad (133)$$

for a polytropic stellar model. Here $B_{12} = B/10^{12}$ G. By comparing this to the timescale of r -mode growth (65) we see that magnetic braking will only be competitive for neutron stars with $B \sim 10^{15}$ G and higher. In this context, Ho and Lai make an interesting observation. They note that, in absence of radiation, the canonical angular momentum is conserved. This means that if we imagine a situation where the star is spun down (or up) by some other agent,

then the amplitude of the mode must change in order to keep J_c constant. Hence, one would expect to find $\alpha \sim 1/\sqrt{\Omega}$. This would not be important under normal circumstances, but it could be highly relevant if other spin-down torques dominate, eg. for a young magnetar.

As we have already discussed in Section 3.1, a mode may be driven unstable by any radiative mechanism. Hence, it is interesting to try to quantify the extent to which electromagnetic radiation from the *r*-modes may affect the growth rate of the instability. Ho and Lai estimate this effect, and based on their results we find

$$t_{\text{em}} \approx -4.8 \times 10^{10} M_{1.4} B_{12}^{-2} R_{10}^{-6} P_{-3}^4 \text{ s.} \quad (134)$$

Clearly, this effect would be relevant only for very strong magnetic fields. Basically, the electromagnetic driving of the mode would become competitive with gravitational radiation for $B \approx 10^{15}$ G, so again we deduce that the effect could be of importance for magnetars. Finally, Ho and Lai also point out that the shaking of the magnetic field lines due to the mode-oscillations will generate Alfvén waves in the magnetosphere. These can also drive the mode, and one can estimate that

$$t_A \approx -2 \times 10^8 M_{1.4} R_{10}^{-2} B_{12}^{-2} \text{ s.} \quad (135)$$

Interestingly, this leads to a more significant driving of an unstable mode than direct electromagnetic radiation (134).

The above results suggest that the magnetic field may have great effect on the *r*-modes in magnetars, i.e. for stars with magnetic fields above (say) 10^{14} G. For stars with weaker magnetic fields, such as recycled neutron stars in LMXBs the picture is not so clear. The outcome depends crucially on whether the *r*-mode leads to differential rotation/drift in the fluid or not. There is also the obvious caveat regarding neutron stars with a solid crust. It could be that the magnetic field is essentially frozen into the crust and that waves in the fluid core do not significantly penetrate the crust. If this is the case, the magnetic field may not have a large effect on the *r*-modes at all. On the other hand, it is expected that the core fluid will be electromagnetically locked into corotation with the crust on a very short time-scale. This may then prevent oscillations in the core entirely. Much further work into the interplay between neutron star pulsation and the magnetic field is needed before we can claim to have even a rudimentary understanding of this problem.

7.4. Nonlinear effects

It is clear that most of the results we have discussed in this article may be strongly affected by nonlinear effects. Nonlinear stellar pulsation theory is a relatively unexplored research area that provides many conceptual and technical challenges. Still, if we want to understand the actual evolution of an unstable pulsation mode and possible effects it may have on the bulk rotation of a star we must explore the non-linear regime.

In the case of the r -mode scenarios discussed in Section 5, one main unknown parameter is the saturation amplitude α_s . The values we used in our discussion ($\alpha_s = 0.01 - 1$) were chosen ad hoc, by assuming that the r -mode would be able to grow to a “large” amplitude. But it is not at all clear that this will be possible. Several nonlinear effects may set in at considerable smaller values of α and prevent further mode-growth. If this is the case, the various astrophysical scenarios must be altered accordingly. Some candidates for nonlinear mode saturation are: i) coupling to other pulsation modes, that could lead to a cascade of energy from the unstable mode into other modes, ii) turbulence, which is likely to play a role since the Reynolds number is very high in the r -mode problem, and iii) nonlinear effects that change the nature of the mode in such a way that, for example, shocks develop. At the present time, the only available results regarding the nonlinear mode saturation follow from an estimate of turbulence that arises in the Ekman layer at the base of the crust¹¹⁷. These estimates suggest that the shear motion in the Ekman layer becomes turbulent when the mode reaches an amplitude**

$$\alpha_c \approx 1.6 \times 10^{-3} \Delta^{-1} T_8^{-1} P_{-3}^{1/2} \quad (136)$$

where $\Delta = |\Delta v|/|v|$ as in Section 4.5. We recall that a typical value might be⁸⁵ $\Delta \approx 0.1$. Hence turbulence would play a role provided that the mode can grow to an amplitude $\alpha \approx 0.1$. Since the turbulent dissipation rate increases as the cube of the amplitude while the gravitational-radiation growth scales as α^2 one can estimate the the mode will saturate when¹¹⁷

$$\alpha_s \approx 3.5 \times 10^{-3} \Delta^{-3} P_{-3}^{-5} \left(\frac{5 \times 10^{-3}}{C_D} \right) \quad (137)$$

where C_D is the relevant drag coefficient. From this result we can deduce that turbulence may saturate the r -mode growth at amplitudes significantly below unity provided that Δ is relatively large. But if $\Delta \approx 0.1$ we get $\alpha_s \approx 3.5$ and it seems likely that other nonlinear mechanisms will provide the mode-saturation.

A second issue of utmost importance regards the spin-evolution of the star during the instability phase, i.e. the back-reaction of the mode onto the bulk rotation of the fluid. In the phenomenological spin-evolution model discussed in Section 5.1 it was assumed that, once the r -mode had reached the saturation amplitude, all the excess angular momentum was drained from the bulk rotation. That this will actually be the case is far from clear, and would in fact seem rather unlikely. Nonlinear effects may well cause differential rotation in the

*It should be noted that the turbulence estimates are based on experimental data for the flow of water above rugged surfaces. It is not clear to what extent such results will be relevant for neutron star oscillations. One should also note that it is assumed that one can meaningfully use the standard r -mode gravitational wave estimates even in the turbulent case. This corresponds to assuming that the turbulence is confined to the Ekman layer, which may be dubious since turbulence is by its very nature convective.

star (or a secular drift as in the study of Rezzolla et al¹³⁰), and if this is the case our current modelling will be inadequate in many ways. A first attempt to gain insight into the non-linear back-reaction has been made by Levin and Ushomirsky¹³². The considered the toy-problem of an *r*-mode in a shell, and find that the *r*-modes do indeed generate differential rotation in the star.

It seems to us that we need numerical studies using fully nonlinear hydrodynamics to investigate these difficult issues. This is obviously a very challenging task. Ideally, one would want to study the onset of instability and follow the mode through to saturation and then see what effect the instability has on the spin of the star. Computationally, this means that one would need a numerical evolution that resolves the mode-oscillation (on a millisecond timescale) and tracks the star through hundreds of seconds. This is certainly not possible given current technology. As we understand it, the best one can hope for at present is to study the star over (say) a few tens of rotation periods. This will not allow a complete study of the problem, but it may still provide answers to many crucial questions. Efforts in this direction are underway, based on either direct numerical evolution of the general-relativistic hydrodynamical equations on a fixed background spacetime^{133,134} or as an integration of the Newtonian hydrodynamical equations and the use of an "accelerated" gravitational radiation reaction force¹³⁵. As we were finishing this review the first results of these efforts became available. Simulations due to Stergioulas and Font indicate that the *r*-modes may be able to grow to a surprisingly large amplitude. In fact, there are no significant signs of mode-saturation until at unrealistically large amplitudes. These results are very promising, and it seems likely that continued efforts in this direction will provide important insights into the nonlinear physics of the *r*-mode instability.

8. Concluding remarks

In this review we have discussed the *r*-modes in rotating neutron stars and the associated gravitational-wave driven instability. This is a research area that has attracted considerable interest following the prediction that hot young neutron stars may be spun down to a level comparable to that extrapolated from observations and the suggestion that the associated gravitational waves may be detectable with the new generation of interferometers.

Our aim in writing this review was to summarize most of the ideas and suggestions that have been made regarding the *r*-mode instability and put them in the appropriate context. In doing this we obviously had to make sacrifices, but hopefully we have managed to provide a useful introduction to the many relevant issues. What should be clear is that our present understanding of instabilities in rotating neutron stars and their potential astrophysical relevance is unsatisfactory in many respects. All current suggestions are based on simplified, often phenomenological, models. The hope must be that future investment

in this field will lead to a truly quantitative description of “realistic” neutron stars, and that the various proposed astrophysical scenarios will be described in detail rather than “order of magnitude”. We have far to go before we reach this goal, and it may well be that actual observations will beat us theorists to the answers of the many challenging questions that we are just beginning to formulate.

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